2

Electromagnetic Waves

2.1 The Nature of Light

Light behaves like a wave phenomenon, but in other respects it acts like a stream of high-speed, submicroscopic particles. In this section we are concerned with the properties of light that are best understood by considering light to be a wave-like phenomenon.

When characterising a wave we describe its height or amplitude and its wavelength as shown in Figure 2.1. If the wave in Figure 2.1 were propagating with velocity v then at a fixed point the number of wavelengths that would pass per unit time i.e. the frequency of the wave is

$$\nu = v/\lambda \tag{2.1}$$

The number of waves per unit distance is the spatial frequency or wavenumber, $\tilde{\nu}$, and is defined

$$\tilde{\nu} = 1/\lambda \tag{2.2}$$

2.1.1 Electromagnetic Spectrum

The regions of the electromagnetic spectrum are not well defined.

2.2 Wave Equation

A wave is a displacement in a medium or a field. Waves that remain in one place are called standing waves. Waves that are moving are called travelling waves, and have a disturbance that varies both with time and location. Travelling waves are important because their motion transports energy. Figure 2.3 shows an example of a travelling wave. The amount of energy that flows per second across a unit area perpendicular to the direction of travel is called the intensity of the wave. If the wave flows continuously there is an energy density such that the intensity is given by the product of the wave velocity and the energy density.

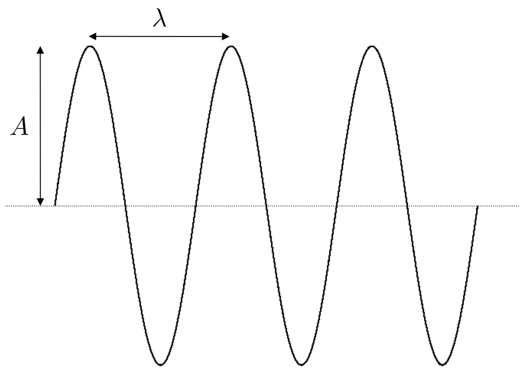


FIGURE 2.1
A sinusoidal wave.

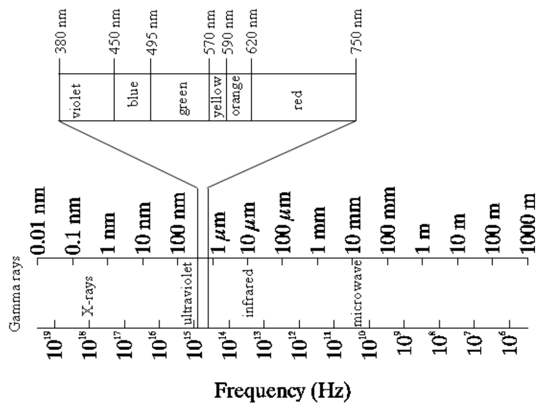


FIGURE 2.2
Electromagnetic spectrum.

TABLE 2.1
Naming of the Electromagnetic Spectrum

Wavelength	Name	Comment
	ultra violet	the region beyond visual perception
315-400 nm	ultra violet A	the region beyond visual perception
280-315 nm	ultra violet B	the region beyond visual perception
< 280 nm	ultra violet C	the region strongly absorbed in the atmosphere and undetectable at the ground
0.7μm	visible	the region of human visual perception
0.7- 1000 μm	infrared	
0.7- 3.7 μm	near-infrared	the region beyond visual perception where the Sun's radiance at 1. A.U. is stronger than the radiance from the Earth
0.7- 1000 μm	mid-infrared	
0.7- 1000 μm	far-infrared	

2.2.1 Waves in One Dimension

Let $f(z, t)$ describe the perturbation from equilibrium of some quantity as a function of location z and time t. For example $f(z, t)$ could represent the displacement of a string held between two points where the displacement is measured in a direction orthogonal to the line of the string. The classical wave equation

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{2.3}$$

describes waves propagating with velocity v in one spatial dimension (z). The general solution of this differential equation is of the form

$$f(z, t) = h(z - vt) + g(z + vt) \tag{2.4}$$

where $h(z - vt)$ represents a wave of shape $h(z)$ (at time $t = 0$) travelling at constant speed v in the positive z direction. The second term, $g(z + vt)$, represents a wave of shape $g(z)$ (at time $t = 0$) travelling in the negative z direction at constant speed v.

An important solution to the wave equation is a sinusoidal wave described by

$$f(z, t) = A_0 \cos(kz - \omega t + \delta) \tag{2.5}$$

where A_0 is the amplitude of the wave. The amplitude is the maximum value that $f(z, t)$ can take while the argument of the cosine function is referred to as the phase, ϕ, of the wave so

$$\phi = kz - \omega t + \delta \tag{2.6}$$

where k is angular wavenumber and ω is the angular frequency of the wave. The phase offset δ is included in Equation 2.5 to account for the phase at the origin at $t = 0$.

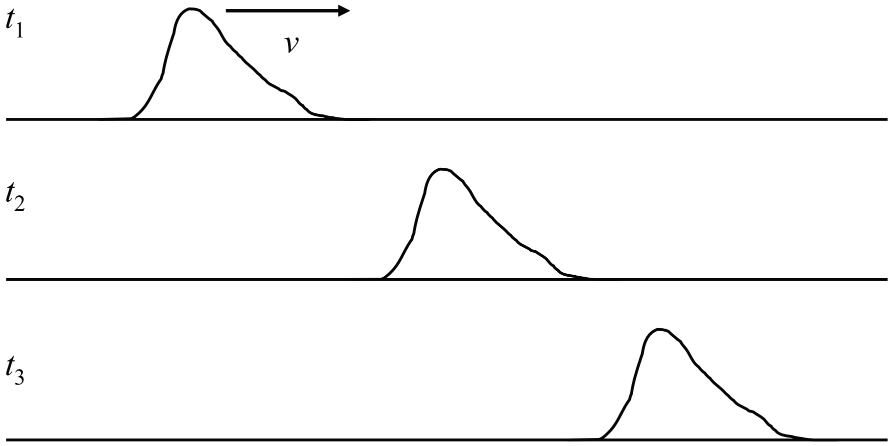


FIGURE 2.3
A travelling wave at three different times t_1, t_2 and t_3 such that $t_3 > t_2 > t_1$.

The ratio of the angular wavenumber and the angular frequency defines the phase velocity v , i.e.

$$v = \frac{\omega}{k}. \tag{2.7}$$

The phase velocity denotes the speed of propagation of a point on the wave. As $\cos\theta = \cos(-\theta)$ a harmonic wave travelling in the positive z direction can be represented by

$$f(z,t) = A_0 \cos(kz - \omega t + \delta) \tag{2.8}$$

or by the conjugate form

$$f(z,t) = A_0 \cos(-kz + \omega t - \delta) \tag{2.9}$$

In this text choice has been made to use the form of equations 2.8.

2.2.2 Waves in Three Dimension

When expressed in three dimensions the wave equation is

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{2.10}$$

where v is the wave speed. The is equation is usually expressed more succinctly by using the ∇^2 operator

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{2.11}$$

giving

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{2.12}$$

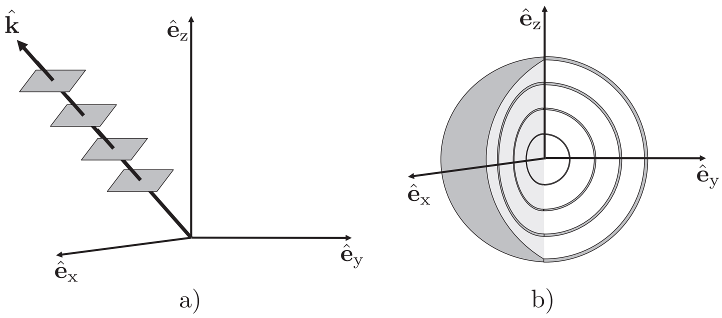


FIGURE 2.4
Stylized view of a) a plane wave propagating in direction $\hat{\mathbf{k}}$ and b) a spherical wave.

This equation was two important solutions:a plane wave and a spherical wave.

A plane wave such as the one shown in Figure 2.4 is a wave where the surfaces of constant phase are infinite parallel planes normal to the direction of propagation and is described by

$$f(r, t) = A_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta) \tag{2.13}$$

where $\mathbf{k} = k\hat{\mathbf{k}}$ is the wave vector (k being the wavenumber). The position vector $\mathbf{r}$ is defined as

$$\mathbf{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z \tag{2.14}$$

The wave function used so far has been a scalar function of location and position. This is adequate to describe, for instance, a pressure wave. However if the wave describes a field strength where the field has a direction then the amplitude is itself a vector.

If the choice of axes and time is arbitrary then it is often convenient to chose the origin and start time to ensure $\delta = 0$, so that A_0 is entirely real, and, to define the z axis as the direction of propagation so that wave amplitudes are in the $x - y$ plane. Adopting these choices gives the plane wave expression as

$$\mathbf{f}(z, t) = \mathbf{A}_0 \cos(kz - \omega t). \tag{2.15}$$

A spherical wave has a constant phase on a sphere of any given radius, r, at a given time, t, so that the solution to the wave equation is

$$\mathbf{f}(r, t) = \frac{\mathbf{A}_0}{r} \cos(kr - \omega t). \tag{2.16}$$

In these cases it is usually easiest to use spherical polar coordinates so the amplitude of the wave is on the $\theta - \phi$ surface. Unlike plane waves the effective amplitude of spherical waves is not constant but decreases as $1/r$.

2.2.3 Complex Representation of a Harmonic Wave

It is straightforward to show that

$$f(z,t) = A_0 e^{i(kz-\omega t+\phi)} \tag{2.17}$$

is a solution of the wave equation where A_0 is the amplitude and $kz - \omega t + \phi$ is the phase of the wave. At any given time the wave amplitude is given by the real part of this expression. The phase offset can be included in the amplitude

$$f(z,t) = \tilde{A}_0 e^{i(kz-\omega t)} \tag{2.18}$$

where the complex amplitude $\tilde{A}_0$ is

$$\tilde{A}_0 = A_0 e^{i\phi} \tag{2.19}$$

The energy carried by a wave is proportional to the amplitude squared, $|A_0|^2$, which is the same as $A_0 A_0^*$ or $\tilde{A}_0 \tilde{A}_0^*$ where the asterisk represents the complex conjugate.

2.2.4 Principle of Superposition

To understand what happens when two (or more) waves occur at the same time we invoke the principle of superposition - the displacement of any point due to the superposition of wave systems is equal to the sum of the displacements of the individual waves at that point. This is shown mathematically using two solutions to the wave equation, $f_1(\mathbf{r}, t)$ and $f_2(\mathbf{r}, t)$. Then

$$\nabla^2(f_1 + f_2) = \nabla^2 f_1 + \nabla^2 f_2 = \frac{1}{v^2}\frac{\partial^2 f_1}{\partial t^2} + \frac{1}{v^2}\frac{\partial^2 f_2}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 (f_1 + f_2)}{\partial t^2} \tag{2.20}$$

It is important to realise that it is the amplitudes of the waves that are being combined. So what happens to the energy when, for example the waves combine to cancel out? In this case energy is not conserved locally as the interference between the waves shifts the spatial distribution of energy. Energy is conserved by considering the spatial domain as a whole.

An arbitrarily shaped wave can be re-expressed in terms of sinusoidal basis functions, i.e. as an integral of sinusoidal functions multiplied by some coefficients. The solution for an arbitrary function is formed by combing the solution for each sinusoidal component. This idea is shown graphically in Figure 2.5. For this reason a description of the propagation of waves is usually limited to harmonic waves, knowing that the general problem can be represented by decomposing an arbitrary shaped wave into a harmonic wave sum.

Waves that are composed of a single frequency are called monochromatic. No physical source is truly monochromatic but many are very close so it is a useful mathematical abstraction. Light can also be described as coherent or incoherent. Coherent radiation emitted by a source occurs when all the elementary waves emitted have a phase difference constant in space and time. Laser light, for instance, can be confined to extremely narrow spectral range and is coherent. Naturally emitted radiation by processes in the Sun's or Earth's atmosphere can also be monochromatic but this radiation is generally incoherent.

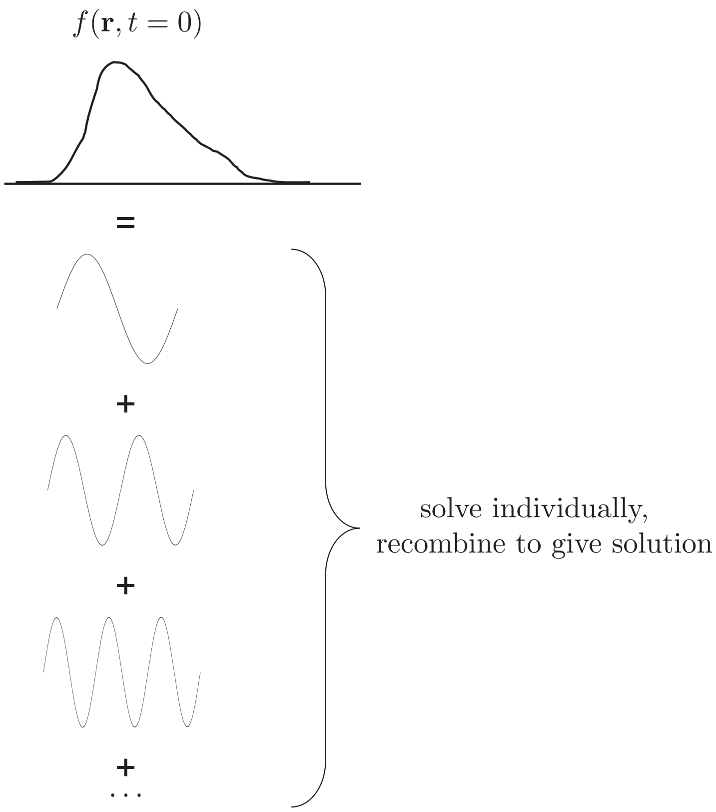


FIGURE 2.5
An arbitrary shaped wave can be decomposed into the sum of sinusoids. Thus finding the wave solution for an arbitrary shaped wave is converted into finding the solution for a sinusoid of arbitrary frequency.

2.3 Light Waves in a Vacuum

The electromagnetic state in a vacuum can be specified by two vectors, the electric field **E** and the magnetic field **H**. If the fields do not change with time then they do not interact. If the fields change with time then they influence each other following Maxwell's equations for a vacuum. These relationships are

$$\nabla \cdot \mathbf{E} = 0 \tag{2.21}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{2.22}$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{2.23}$$

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.24}$$

where μ_0 and ϵ_0 are respectively the permeability and permittivity of the vacuum.

To decouple the electric and magnetic components of these equations it is useful to introduce the vector identity

$$\nabla \times (\nabla \times \mathbf{r}) = \nabla(\nabla \cdot \mathbf{r}) - \nabla^2(\mathbf{r}) \tag{2.25}$$

for some vector $\mathbf{r}$. Using this relation gives

$$\nabla \times (\nabla \times \mathbf{E}) = \left[\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}\right] = -\nabla^2 \mathbf{E} \tag{2.26}$$

Substituting in Equation 2.23 gives

$$\nabla^2 \mathbf{E} = -\nabla \times \left(-\mu_0 \frac{\partial \mathbf{H}}{\partial t}\right) = \mu_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) \tag{2.27}$$

Substituting in Equation 2.24 gives

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{2.28}$$

Similarly the curl of Equation 2.24 gives

$$\nabla \times (\nabla \times \mathbf{H}) \left[= \nabla(\nabla \mathbf{H}) - \nabla^2 \mathbf{H} = -\nabla^2 \mathbf{H}\right] = -\nabla \times \left(-\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) = \epsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{E}) \tag{2.29}$$

Substituting in Equation 2.23 gives

$$\nabla^2 \mathbf{H} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{H}}{\partial t^2}. \tag{2.30}$$

Both Equation 2.28 and 2.30 are of the form of the wave equation derived in Section 2.2 with a phase speed of $v = c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ i.e. the speed of light. The plane wave solutions are

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \tag{2.31}$$

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta)} \tag{2.32}$$

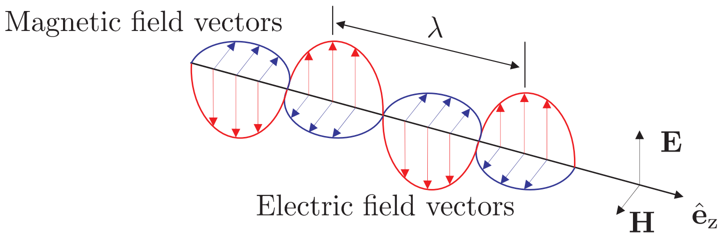


FIGURE 2.6
Electromagnetic wave.

where δ is the phase offset between the electric and magnetic waves. Substituting these expressions into Maxwell's equations shows that δ must be 0 and

$$\mathbf{k} \times \mathbf{E}(\mathbf{r}, t) = \mu_0 \omega \mathbf{H}(\mathbf{r}, t) \tag{2.33}$$

i.e. that **k**, **E** and **H** are mutually perpendicular vectors and $E(\mathbf{r}, t) = cB(\mathbf{r}, t)$ as $c = \omega/k$ and $\mathbf{B}(\mathbf{r}, t) = \mu_0 \mathbf{H}(\mathbf{r}, t)$. Figure 2.6 depicts the resulting electromagnetic wave.

2.4 Light Waves in Matter

2.4.1 Matter & Polarizability

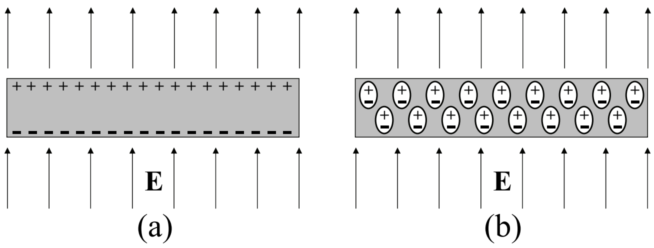


FIGURE 2.7
Representation of the application of an electric field to a) a conductor and b) an insulator.

Although matter comes in many varieties — solids, metals, glasses, liquids gases — most material can be classed as either a conductor or insulator (sometimes called a dielectric). In a metallic conductor electrons are free to move within the material while in a liquid conductors it is ions within the fluid that move. When an electric field is applied to a conductor the electrons (or ions) move to form a surface charge and the net electric field within the conductor is zero. In an insulator each electron is attached to a particular atom and the application of an electric field may distorts their location. This is shown schematically in Figure 2.4.1.

The movement of charge within a dielectric is captured in the concept of an electric dipole, **p** is defined as the spatial-mean charge. For a discrete set of n charges $q_1, q_2, q_3 \ldots q_n$ at locations $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \ldots \mathbf{r}_n$ the electric dipole is

$$\mathbf{p} \equiv \sum_{i=1}^{n} q_i \mathbf{r}_i. \tag{2.34}$$

For a continuous charge-distribution ρ

$$\mathbf{p} \equiv \iiint \mathbf{r}\rho(\mathbf{r})\, dx\, dy\, dz. \tag{2.35}$$

Some molecules have an asymmetric charge distribution that gives rise to a permanent electric dipole. Examples of polar molecules are H_2O, CO, NH_3 and HF. Under influence of an electric field the charge-distribution can distort and change the strength of the electric dipole. The strength of the induced dipole **p** is a function of the applied electric field as

$$\mathbf{p} = \alpha\, \mathbf{E}. \tag{2.36}$$

where α is a second rank Cartesian tensor called the polarizability. The polarizability describes the ease by which a material can be polarized under the influence of an external electric field **E**. If **p** is parallel to **E** then α reduces to a scalar.

The value of α and its units depends upon the system and definition adopted. If Equation 2.36 is in SI units then the units of α are $\mathrm{C\,m^2\,V^{-1}}$. The units for α become cm^3 in the cgs system. Finally if the polarizability α' is defined through the expression

$$\mathbf{p} = 4\pi\epsilon_0 \alpha'\, \mathbf{E} \tag{2.37}$$

then it has units of m^3.

2.4.2 Maxwell's Equations for Light in a Medium

When fields are present in a material they act on the electrons and ions to induce electric and magnetic dipole moments. Generally a material is neutral but in response to the applied field the charge distribution may shift creating dipole moments. Currents induced in the material can generate magnetic dipole moments. If the applied fields vary with time the induced electric and magnetic dipoles will generate electromagnetic waves that interact with the incident wave and alter its propagation characteristics. When an electromagnetic wave travels through a material the oscillating electric field sets some of the electrons in the medium into forced vibration. The vibrating electrons will generate new waves of their own. If the vibrating electrons are sufficiently close together they will be driven coherently. The scattered wave can be superimposed with the incident wave to give rise to the wave in the material. If the electrons are far apart compared with the wavelength of the radiation the electrons are driven incoherently and a scattered wave results.

In matter the electromagnetic state is described by four macroscopic quantities:

- ρ — he volume density of electric charge,
- **P** — the volume density of electric dipoles,
- **J** — the current density i.e. the electric current per unit area,
- **M —** the volume density of magnetic dipoles.

These quantities are considered to be averaged over a volume to eliminate the variations due to the atomic structure of matter. Consider microscopic form of Maxwell's equations in terms of the macroscopic fields **E**, **B** and the macroscopic charge density ρ and current density **J**.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} - \frac{1}{\epsilon_0}\nabla \cdot \mathbf{P} \tag{2.38}$$

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \tag{2.39}$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} - \mu_0 \frac{\partial \mathbf{M}}{\partial t} \tag{2.40}$$

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} + \mathbf{J} \tag{2.41}$$

To understand the behaviour of the wave in a medium Maxwell's equations must be supplemented by the material equations which describe the behaviour of substances under the influences of a field. Dielectric materials become polarised in an electric field with the result that electric field is greater than it would be in free space. Therefore it is useful to define the electric displacement **D** as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{2.42}$$

and the magnetic induction, **B** as

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{2.43}$$

These definitions allow Equations 2.38 to 2.41 to be expressed in a more compact form i.e. **E**, **B** and the macroscopic charge density ρ and current density **J**.

$$\nabla \cdot \mathbf{D} = \rho \tag{2.44}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.45}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.46}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{2.47}$$

2.4.3 Maxwell's Equations in a Non-magnetic, Neutral Medium

Most media encountered in the atmosphere are non-magnetic and are electrically neutral so that both **M** and ρ are zero. Equations 2.38 to 2.41

$$\nabla \cdot \mathbf{E} = -\frac{1}{\epsilon_0}\nabla \cdot \mathbf{P} \tag{2.48}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{2.49}$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{2.50}$$

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} + \mathbf{J} \tag{2.51}$$

A wave equation solution for the electric field can be derived following the same steps as in Section 2.3, i.e. taking the curl of Equation 2.50 gives

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\mu_0 \frac{\partial \mathbf{H}}{\partial t}\right) \tag{2.52}$$

$$\Rightarrow \nabla^2 \mathbf{E} - \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} + \mu_0 \frac{\partial \mathbf{J}}{\partial t} \tag{2.53}$$

The two terms on the right hand side of Equation 2.53 are due to the presence of polarization charge and conduction charge respectively. For non-conducting media J = 0 and only the polarization term is important. For strongly conducting media the conduction charge term dominates. Here we only consider the propagation of light in an isotropic dielectric. Light propagation in metals is covered by, for instance, *Fowles* [1989].

2.4.4 Light Waves in a Linear Medium

In general **B** (or **D**) is not even a unique function of **H** (or **E**), but depends upon the earlier time evolution (hysteresis). If the field the material properties are isotropic and linear with respect to the imposed field so that

$$\mathbf{J} = \sigma \mathbf{E} \tag{2.54}$$

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 (1 + \chi) \mathbf{E} \tag{2.55}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_0 (1 + \chi_m) \mathbf{H} \tag{2.56}$$

where σ is the specific conductivity, ϵ is the permittivity, χ is the electric susceptibility and μ is the magnetic permeability and χ_m is the magnetic susceptibility.

Inserting the linear expression for displacement (Equation 2.55) into Equation 2.42 allows the polarization to be written as a function of the applied electric field i.e.

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} = (\epsilon - \epsilon_0) \mathbf{E} \tag{2.57}$$

[redacted] [redacted] In the absence of any free charge or free currents equations XX to XX simplify further to Maxwell's equation become:

$$\nabla \cdot \mathbf{E} = 0 \tag{2.58}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.59}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.60}$$

$$\nabla \times \mathbf{B} = \mu\epsilon\frac{\partial \mathbf{E}}{\partial t} \tag{2.61}$$

These are the same as the Maxwell equations in a vacuum except that $\mu_0\epsilon_0$ has been replaced with $\mu\epsilon$. This implies that electromagnetic waves propagate through a linear medium at a speed $v = \frac{1}{\epsilon_0\mu_0}$. The ratio the speed of light in a vacuum to the speed of light in a medium is the index of refraction n. For most materials μ is very close to μ_0 so

$$\tilde{n} = n + i\kappa = \frac{c}{v} = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \approx \sqrt{\frac{\epsilon}{\epsilon_0}} \tag{2.62}$$

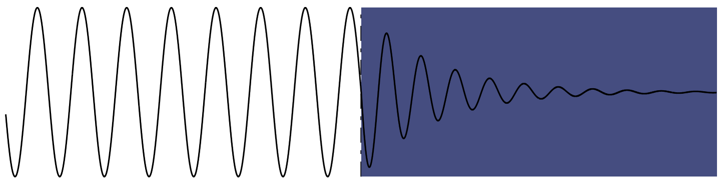


FIGURE 2.8
Decay of an electromagnetic wave as it enters a electrically insulating, non-magnetic material.

In an electrically insulating, non-magnetic material, the plane wave solution of Maxwell's equations in a medium of permittivity ϵ is

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{2.63}$$

Substituting $\mathbf{k} = \frac{\omega}{c}(n + i\kappa)\hat{\mathbf{k}}$ into the wave equation gives

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i[\omega(n/c)\hat{\mathbf{k}}\cdot\mathbf{r} - \omega t]} e^{-\omega(\kappa/c)\hat{\mathbf{k}}\cdot\mathbf{r}} \tag{2.64}$$

This represents a decaying wave where κ determines the absorption and n the phase velocity. The situation is illustrated in Figure 2.8. The skin depth, δ_{skin}, is the distance a wave penetrates before its amplitude is reduced by $1/e$, hence

$$\delta_{\text{skin}} = \frac{c}{\omega\kappa} = \frac{\lambda}{2\pi\kappa} \tag{2.65}$$

The penetration depth, $\delta_{\text{penetration}}$, is the distance a ray travels in an absorbing medium before its energy has been reduced by a factor of $1/e$. As the energy of a wave is proportional to amplitude squared it follows

$$\delta_{\text{penetration}} = 2\delta_{\text{skin}} \tag{2.66}$$

For light with a wavelength of 0.55 μm the penetration depth varies from ~ 3.5 μm for $\kappa = 0.1$ to ~ 350 μm for $\kappa = 0.001$.

Finally confusion can arise if an electromagnetic wave is represented by

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} \qquad \text{[conjugate form]} \tag{2.67}$$

The corresponding representation of refractive index is $\tilde{n} = n - i\kappa$ to ensure the wave decays in the direction of propagation in an absorbing medium.

2.4.5 The Dielectric Sphere Model

A simple model of an atom, of radius a, consists of a positive nucleus (charge $+q$) surrounded by a uniformly charged spherical cloud (total charge $-q$). The electric field a distance r from the centre of a uniformly charged sphere is [*Duffin*, 1990]

$$\mathbf{E}_{\text{int}} = \frac{q\mathbf{r}}{4\pi\epsilon_0 a^3} \tag{2.68}$$

If an electric field E is applied then the nucleus will be displaced a distance d such that the external force from the field will cancel the internal force from the charge distribution i.e.

$$\mathbf{E} = \frac{q\mathbf{d}}{4\pi\epsilon_0 a^3} \tag{2.69}$$

which can be rearranged to give

$$\mathbf{p} = q\mathbf{d} = 4\pi\epsilon_0 a^3 \mathbf{E}. \tag{2.70}$$

A simple description of an atom (or non-polar molecule) is of two equal and opposite charges that move apart under the influence of an electric field to create a dipole. This is shown schematically in Figure 2.9. If the molecules are polar then the orientation of any dipoles present will be insignificant for high frequency optical fields [*Kerker*, 1969]. The tendency for charge to separate within an atom or molecule is encapsulated in the polarizability α which is defined as the mean electric dipole moment per unit field, i.e.

$$\mathbf{p} = \alpha\mathbf{E} \tag{2.71}$$

The polarization, $\mathbf{P}$ is defined as the mean dipole moment per unit volume. If an isolated sphere of radius a is illuminated by a beam of linearly polarized light it becomes polarized. The electric potential V at position (r, θ) can be determined using Laplace's equations with appropriate boundary conditions [*Jackson*, 1999] as

$$V_{\text{in}} = -\frac{3}{\epsilon/\epsilon_0 + 2} E_0 r \cos\theta \qquad r \leq a \tag{2.72}$$

$$V_{\text{out}} = -r\cos\theta E_0 + \frac{(\epsilon/\epsilon_0 - 1)}{(\epsilon/\epsilon_0 + 2)} \frac{a^3}{r^2} E_0 \cos\theta \qquad r > a \tag{2.73}$$

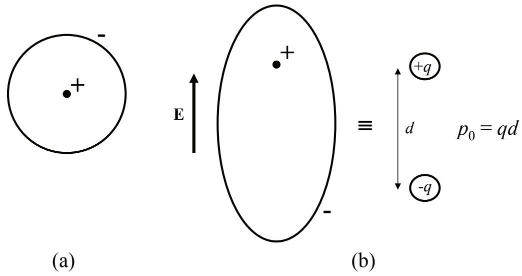


FIGURE 2.9
a) Neutral atom or molecule modelled as central positive charge surrounded by equal and opposite charge. b) Slight separation of charge on application of an electric field and equivalence to a dipole.

where ϵ is the electric inductive capacity of the sphere. The electric field is determined using $\mathbf{E} = -\nabla V$. Inside the sphere this gives

$$\mathbf{E}_{\text{in}} = \frac{3}{\epsilon/\epsilon_0 + 2}\mathbf{E}_0 \tag{2.74}$$

so that the electric field is uniform and parallel to the external field. Using the relationship between polarization and electric field in a linear medium (Equation 2.57) gives

$$\mathbf{P} = (\epsilon - \epsilon_0)\mathbf{E}_0 = 3\epsilon_0\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\mathbf{E}_0 = 3\epsilon_0\frac{\tilde{n}_0^2 - 1}{\tilde{n}^2 + 2}\mathbf{E}_0 \tag{2.75}$$

From which the dipole moment of the a dielectric sphere is

$$\mathbf{p} = \frac{4}{3}\pi a^3\mathbf{P} \tag{2.76}$$

and the polarizability is

$$\alpha = \frac{|\mathbf{p}|}{|\mathbf{E}_0|} = \frac{4}{3}\pi a^3 3\epsilon_0\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2} = 4\pi a^3\epsilon_0\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2} = 4\pi a^3\epsilon_0\frac{\tilde{n}^2 - 1}{\tilde{n}^2 + 2} \tag{2.77}$$

Outside the sphere the electric field given by $\mathbf{E} = -\nabla V$ is

$$\mathbf{E}_{\text{out}} = E_0(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) + \frac{(\epsilon/\epsilon_0 - 1)}{(\epsilon/\epsilon_0 + 2)}\frac{a^3}{r^3}E_0(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}). \tag{2.78}$$

which is the sum of the original field and the field due to the induced dipole moment, **p**.

The resultant electric field lines inside and outside the sphere are shown in Figure 2.4.5. It is important to recognize that the dielectric sphere influences the electric field outside its own physical dimension.

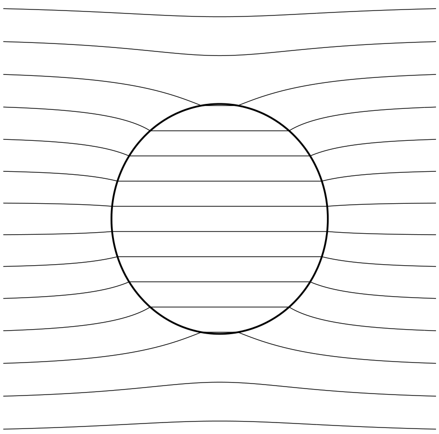


FIGURE 2.10
Resultant electric field near a dielectric sphere in a constant applied field.

2.4.6 The Classical Damped Harmonic Oscillator

In a non-conducting isotropic medium the electrons are permanently bound to the atoms comprising the medium. Consider a macroscopic volume containing N electrons per unit volume. If each electron of charge $-e$ is displaced a distance $\mathbf{r}$ from its equilibrium position the polarization $\mathbf{P}$ of the medium is given by

$$\mathbf{P} = -Ne\mathbf{r} \tag{2.79}$$

In the classical harmonic oscillator model of a molecule the value of the complex refractive index can be found by solving the equation of motion for an electron driven by a force due to the imposed electric field,

$$m\ddot{\mathbf{r}} + m\gamma\dot{\mathbf{r}} + m\omega_0^2\mathbf{r} = -e\mathbf{E} \tag{2.80}$$

where m is the mass of the electron, ω_0 is the natural frequency and γ is the damping coefficient. Equation 2.80 has the solution

$$\mathbf{r} = \frac{-e/m}{\omega_0^2 - \omega^2 - i\omega\gamma}\mathbf{E} \tag{2.81}$$

so that the macroscopic polarization is

$$\mathbf{P} = \frac{e^2N/m}{\omega_0^2 - \omega^2 - i\omega\gamma}\mathbf{E} \tag{2.82}$$

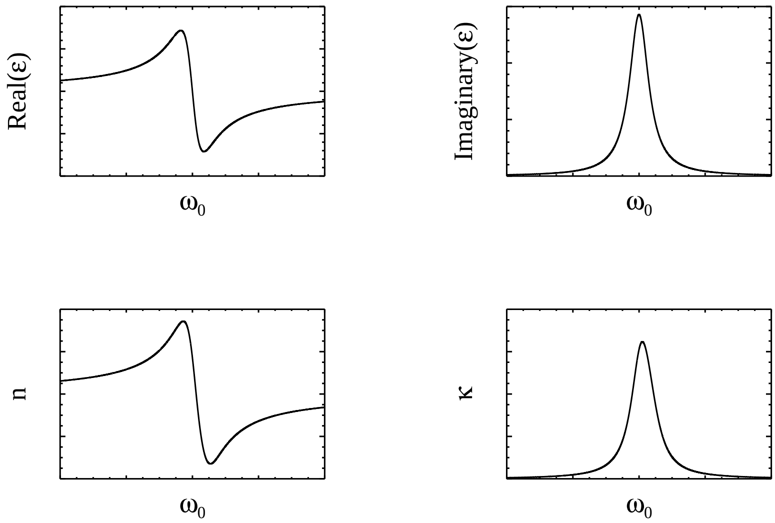


FIGURE 2.11
Change in the values of permittivity (top) and refractive index (bottom) about an absorption band.

From the definitions of susceptibility and permittivity it follows that follows that

$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{Ne^2/(m\epsilon_0)}{\omega_0^2 - \omega^2 - i\omega\gamma} \tag{2.83}$$

Using Equation 2.62 gives

$$(n - i\kappa)^2 = 1 + \frac{Ne^2/(m\epsilon_0)}{\omega_0^2 - \omega^2 - i\omega\gamma} \tag{2.84}$$

Equating the real and imaginary parts gives

$$n^2 - \kappa^2 = 1 + \frac{Ne^2}{m\epsilon_0}\left(\frac{(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}\right) \tag{2.85}$$

$$-2n\kappa = \frac{Ne^2}{m\epsilon_0}\left(\frac{\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}\right) \tag{2.86}$$

from which n and κ can be determined. A plot of the changes in the component of permittivity and refractive index about an absorption band are shown in Figure 2.11. The shapes are very similar ...

This derivation assumed that all the electrons were identically bound. Instead it is possible to assume our macroscopic volume contains N groups of differently

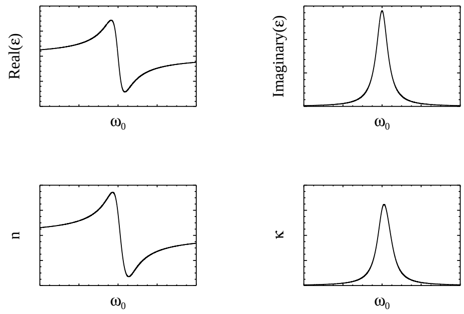


FIGURE 2.12
Wave.Add equilibrium line

bound electrons where the i^{th} group with resonant frequency ω_i and damping constant $\gamma_i$contains a fraction f_i of the total electrons. In this case the polarization is expressed as

$$\mathbf{P} = \frac{e^2 N}{m} \sum_{i=1}^{N} \left(\frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i} \right) \mathbf{E} \tag{2.87}$$

and the corresponding formula from which the complex refractive index can be calculated are

$$m^2 - \kappa^2 = 1 + \frac{Ne^2}{m\epsilon_0} \sum_{i=1}^{N} \left(\frac{f_i}{(\omega_i^2 - \omega^2)^2 + \omega^2\gamma_i^2} \right) \tag{2.88}$$

$$2mk = \frac{Ne^2}{m\epsilon_0} \sum_{i=1}^{N} \left(\frac{\omega\gamma_i f_i}{(\omega_i^2 - \omega^2)^2 + \omega^2\gamma_i^2} \right) \tag{2.89}$$

A plot of the change in refractive index about a collection of absorption bands is shown in Figure 2.12.

The polarizability, α, can be derived from the principle of the dispersion of electromagnetic waves and it is given by

$$\alpha = \frac{3}{4\pi N} \left(\frac{m^2 - 1}{m^2 + 2} \right), \tag{2.90}$$

where N is the total number of molecules per unit volume and $m = n - ik$ is the complex refractive index of the molecules. This equation is called the **Lorentz-Lorenz** formula.*

*The equivalent formula in solid state physics is called the Clausius-Mossotti relation.

2.5 Electromagnetic Wave Energy

The energy density, u, associated with an electric field is the energy per unit volume and in free space is given by [*Duffin*, 1990]

$$u_{\mathrm{E}} = \frac{\epsilon_0 E_0^2}{2} \tag{2.91}$$

The magnetic field contains

$$u_{\mathrm{B}} = \frac{B_0^2}{2\mu_0}. \tag{2.92}$$

These are equal amounts of energy so for an unpolarised electromagnetic wave in a vacuum the total energy density is

$$u = \epsilon_0 E_0^2 = \frac{B_0^2}{\mu_0} \tag{2.93}$$

If the electromagnetic wave is travelling at speed c then in time interval Δt the energy

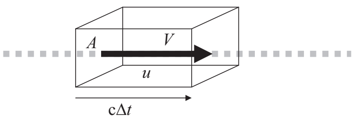


FIGURE 2.13
An electromagnetic wave travelling at speed c and crossing an area A will fill a box of volume $V = c\Delta t$. If the energy density of the wave is u then the energy in the volume is $uc\Delta tA$.

carried across an area A is the product of the volume and energy density as shown in Figure 2.5. The power per unit area associated with the wave is uc. The Poynting vector, $\mathbf{S}$, is defined as the instantaneous energy per unit area per unit time flowing perpendicular to a surface. For a linear medium the Poynting vector is

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \tag{2.94}$$

For a plane harmonic wave this can be rewritten as

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}_0 \times \mathbf{H}_0 \cos^2(\mathbf{k} \cdot \mathbf{r} - \omega t) \tag{2.95}$$

Measurements of radiation are usually over an extended period compared to the period of oscillation of an electromagnetic wave. What is measured is the average magnitude of **S**. The average value of the Poynting vector for a plane harmonic wave is

$$\langle \mathbf{S} \rangle = \frac{1}{2}\mathbf{E}_0 \times \mathbf{H}_0 \tag{2.96}$$

as the average of $\cos^2$ over a period is 1/2. This expression for the power per unit area of a wave is the link between the electromagnetic treatment of light and the radiometric approach. The magnitude of the Poynting vector can be expressed in several ways, e.g. as $H_0 = E_0/c\mu_0$ and $c = 1/\sqrt{\mu_0\epsilon_0}$ in free space then

$$\langle S \rangle = \frac{1}{2}\frac{E_0^2}{c\mu_0} = \frac{1}{2}\epsilon_0 c E_0^2 \qquad \text{[free space]} \tag{2.97}$$

which has the important implication that the rate of energy flow is proportional to the square of the amplitude of the electric field. Also

$$\langle S \rangle = \frac{1}{2}E_0^2\sqrt{\frac{\epsilon_0}{\mu_0}} = \frac{E_0^2}{2Z_0} \qquad \text{[free space]} \tag{2.98}$$

where $Z_0 (= \sqrt{\mu_0/\epsilon_0})$ is the resistance of free space and has a value of 376.6 Ω.

2.6 Polarization

An electromagnetic wave is characterized by electric and magnetic vectors **E** and **H** which form an orthogonal set with the direction of propagation of the wave. In unpolarized light, **E** and **H** have no preferred direction: the waves have electric and magnetic field vectors in random directions in the plane orthogonal to the direction of motion. In polarized light there is a preferred direction for the electric and magnetic field vectors in the wave.

In discussing electromagnetic waves we have assumed a constant amplitude and phase. However a light beam consists of many waves in rapid succession. If the end point of the electric field vector of an unpolarized light beam was viewed from the direction of light propagation the points would randomly fill a circle of radius E_0. When the end point of the electric vector of a polarized light beam is viewed along the direction of light propagation, it moves along a straight line if the light is linearly polarized, along a circle if it is circularly polarized, and along an ellipse if it is elliptically polarized. This idea is shown in Figure 2.14

Most light sources emit unpolarized light, but there are several ways light can be polarized. Processes that give rise to polarization include scattering by air molecules and particles and reflection from the Earth's surface. A full treatment of polarization in radiative modelling is necessary if the observing instrument is sensitive to the polarization state of the observed light. Alternatively instruments can be designed which exploit their sensitivity to polarization state and so provide information on the atmosphere or surface.

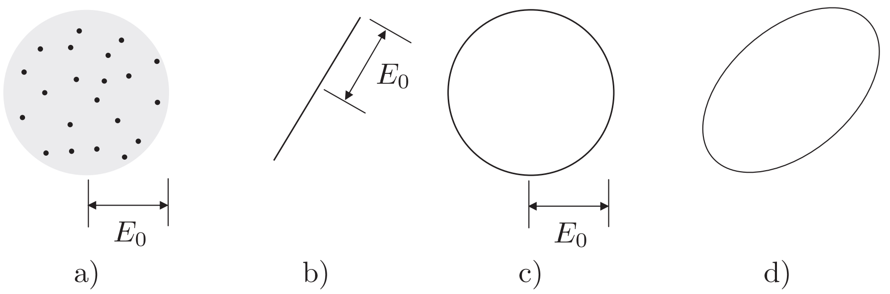


FIGURE 2.14
Representation of the end point of the electric field vector in the plane orthogonal to the direction of propagation. The cases are a)unpolarised, b) linearly polarised, c) circular polarization. d) elliptically polarised.

2.6.1 Mathematical Description of Polarized Light

Consider a polarized electromagnetic wave with propagation constant k and circular frequency ω moving in the z direction. To describe the state of polarization of the wave choose an arbitrary pair of directions at right-angles that lie in the plane orthogonal to the direction of propagation. If the two components are denoted by subscripts b and d respectively then the general elliptically polarized wave is described in terms of the two components by

$$\mathbf{E}(z,t) = E_b(z,t)\hat{\mathbf{e}}_\mathrm{b} + E_d(z,t)\hat{\mathbf{e}}_\mathrm{d} \tag{2.99}$$

where

$$E_b(z,t) = E_{b_0}e^{i(kz-\omega t)} \qquad \text{and} \qquad E_d(z,t) = E_{d_0}e^{i\delta}e^{i(kz-\omega t)} \tag{2.100}$$

where the phase difference between the components, δ, has been shown explicitly so that E_{b_0} and E_{d_0} are real. The amplitudes are given by the real part of each expression so

$$E_b(z,t) = E_{b_0}\cos(kz-\omega t) \qquad \text{and} \qquad E_d(z,t) = E_{d_0}\cos(kz-\omega t+\delta) \tag{2.101}$$

These two formula an be rearranged to give

$$\left(\frac{E_b}{E_{b_0}}\right)^2 + \left(\frac{E_d}{E_{d_0}}\right)^2 + 2\frac{E_b}{E_{b_0}}\frac{E_d}{E_{d_0}}\cos\delta = \sin^2(\delta) \tag{2.102}$$

This is the equation for an ellipse inclined at an angle, χ to the E_b axis as shown in Figure 2.15. This angle is given by

$$\tan 2\chi = \frac{2E_{b_0}E_{d_0}\cos\delta}{E_{b_0}^2 + E_{d_0}^2}$$

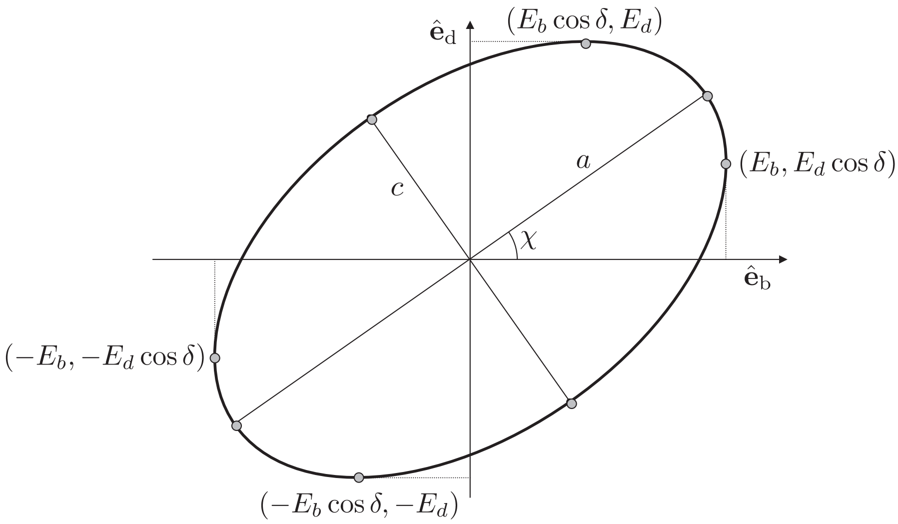


FIGURE 2.15
XXXX Path followed by the tip of the electric field vector whose components have a phase difference of δ. Note that direction of propagation is into the page. If the path taken by the tip of the electric field vector is clockwise then polarisation is called left-handed. Conversely if the electric field vector rotates anti-clockwise the polarisation is called right-handed.

The ellipticity of the ellipse, β is determined from the length of the major and minor axes, c and b respectively, i.e.

$$\tan\beta = \pm c/a \tag{2.103}$$

By convention the ellipticity is positive for right hand polarization and negative for left handed polarization.

Two special cases are important:

Linear polarization If $\delta = m\pi$ (where m is an integer) then Equation 2.102 becomes

$$\left(\frac{E_b}{E_{b_0}} + \frac{E_d}{E_{d_0}}\right)^2 = 0 \tag{2.104}$$

The waves in this case are linearly polarised.

Circular polarization If $\delta = m\pi/2$ (where $m = \pm 1, \pm 3, \ldots$) and $E_{b_0} = E_{d_0} = E_0$ then Equation 2.102 becomes

$$E_d^2 + E_b^2 = E_0^2 \tag{2.105}$$

The waves in this case are circularly polarised.

If only linear optical processes are considered then an outgoing wave represented by two field components $E_{\parallel}^{\text{out}}$ and $E_{\perp}^{\text{out}}$ can be represented by a linear sum of the incoming beam components $E_{\parallel}^{\text{in}}$ and $E_{\perp}^{\text{in}}$ and some associated phase change $(kz - kr)$, i.e.

$$\begin{pmatrix} \mathbf{E}_{\text{l}}^{\text{out}} \\ \mathbf{E}_{\text{r}}^{\text{out}} \end{pmatrix} = \frac{e^{i(kz-kr)}}{ikr} \begin{pmatrix} a_2 & a_3 \\ a_4 & a_1 \end{pmatrix} \begin{pmatrix} \mathbf{E}_{\text{l}_0}^{\text{in}} \\ \mathbf{E}_{\text{r}_0}^{\text{in}} \end{pmatrix} e^{i(kz-\omega t)} \tag{2.106}$$

where the a_1, a_2, a_3, a_4 are amplitude attenuation coefficients associated with the process involved.

2.6.2 Stokes Parameters

The state of polarization is specified by four parameters, two amplitudes and the magnitude and sign of the phase difference. Stokes (1852) introduced the four parameters to describe the elliptically polarized wave, defined

$$I_{\text{inst}} = E_b E_b^* + E_d E_d^* = E_{b_0}^2 + E_{d_0}^2 \tag{2.107}$$

$$Q_{\text{inst}} = E_b E_b^* - E_d E_d^* = E_{b_0}^2 - E_{d_0}^2 \tag{2.108}$$

$$U_{\text{inst}} = E_b E_d^* + E_d E_b^* = 2E_{b_0} E_{d_0} \cos\delta \tag{2.109}$$

$$V_{\text{inst}} = i\left[E_b E_d^* - E_d E_b^*\right] = 2E_{b_0} E_{d_0} \sin\delta \tag{2.110}$$

where the asterisk denotes the conjugate complex value and the subscript i denotes the parameters are instantaneous values. There are only three independent quantities as the parameters are related through

$$I_{\text{inst}}^2 = Q_{\text{inst}}^2 + U_{\text{inst}}^2 + V_{\text{inst}}^2 \tag{2.111}$$

When measuring a light beam, even for a very short time, many electromagnetic waves with independent phases are collected. Consequently measurable intensities are expressed in terms of time averages. The Stokes parameters become

$$I = \langle E_{d_0}^2 \rangle + \langle E_{b_0}^2 \rangle \tag{2.112}$$

$$Q = \langle E_{d_0}^2 \rangle - \langle E_{b_0}^2 \rangle \tag{2.113}$$

$$U = \langle 2E_{d_0} E_{b_0} \cos\delta \rangle \tag{2.114}$$

$$V = \langle 2E_{d_0} E_{b_0} \sin\delta \rangle \tag{2.115}$$

In this case

$$I^2 \geq Q^2 + U^2 + V^2 \tag{2.116}$$

and the degree of polarization of a light beam is defined

$$P = \frac{(Q^2 + U^2 + V^2)^{1/2}}{I} \tag{2.117}$$

For linear polarization then either the $\hat{\mathbf{e}}_{\text{b}}$ or $\hat{\mathbf{e}}_{\text{d}}$ component is zero and $U = V = 0$. Unpolarized light is characterised by the same electric field in the perpendicular and parallel directions and by a random phase relation between the two components. Natural light is unpolarized and can be viewed as the incoherent sum of two beams polarised at right angles. So assuming these polarisation directions to be $\hat{\mathbf{e}}_{\rm b}$ and $\hat{\mathbf{e}}_{\rm d}$ gives $Q = U = V = 0$ and $\langle E_{b_0}^2 \rangle = \langle E_{d_0}^2 \rangle$.

2.7 Reflection and Transmission

Consider a plane harmonic wave incident on a boundary between two optical media that gives rise to reflected and transmitted waves, i.e.

$$\mathbf{E}^{\rm i}(\mathbf{r},t) = \mathbf{E}_0^{\rm i} e^{i(\mathbf{k}^{\rm i}\cdot\mathbf{r}-\omega t)} \qquad \text{incident} \qquad (2.118)$$

$$\mathbf{E}^{\rm r}(\mathbf{r},t) = \mathbf{E}_0^{\rm r} e^{i(\mathbf{k}^{\rm r}\cdot\mathbf{r}-\omega t)} \qquad \text{reflected} \qquad (2.119)$$

$$\mathbf{E}^{\rm t}(\mathbf{r},t) = \mathbf{E}_0^{\rm t} e^{i(\mathbf{k}^{\rm t}\cdot\mathbf{r}-\omega t)} \qquad \text{transmitted} \qquad (2.120)$$

The magnetic field vectors follow from Maxwell's equations as

$$\mathbf{H}^{\rm i}(\mathbf{r},t) = \frac{1}{\mu\omega}\mathbf{k}^{\rm i} \times \mathbf{E}_0^{\rm i} e^{i(\mathbf{k}^{\rm i}\cdot\mathbf{r}-\omega t)} \qquad \text{incident} \qquad (2.121)$$

$$\mathbf{H}^{\rm r}(\mathbf{r},t) = \frac{1}{\mu\omega}\mathbf{k}^{\rm r} \times \mathbf{E}_0^{\rm r} e^{i(\mathbf{k}^{\rm i}\cdot\mathbf{r}-\omega t)} \qquad \text{reflected} \qquad (2.122)$$

$$\mathbf{H}^{\rm t}(\mathbf{r},t) = \frac{1}{\mu\omega}\mathbf{k}^{\rm t} \times \mathbf{E}_0^{\rm t} e^{i(\mathbf{k}^{\rm i}\cdot\mathbf{r}-\omega t)} \qquad \text{transmitted} \qquad (2.123)$$

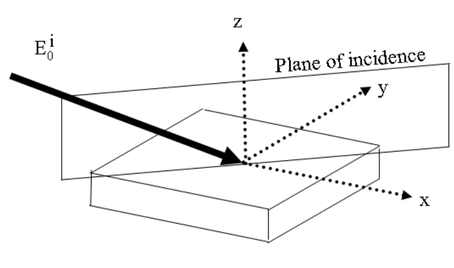


FIGURE 2.16
.

As the energy reflected at a boundary does not vary with position along the boundary or with time there must be a fixed relationship between the incident and reflected wave amplitudes. Similarly the transmitted energy does not vary so similar conditions apply.

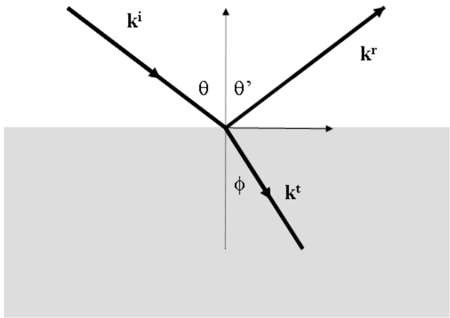


FIGURE 2.17
.

For the waves to be in phase at the boundary

$$\mathbf{k}^{\mathrm{i}} \cdot \mathbf{r} = \mathbf{k}^{\mathrm{r}} \cdot \mathbf{r} = \mathbf{k}^{\mathrm{t}} \cdot \mathbf{r} \tag{2.124}$$

which implies the waves all lie in the same plane which is called the plane of incidence and is shown in Figure 2.16. Evaluating the terms in Equation 2.124 using the angles defined in Figure 2.17 gives

$$k^{\mathrm{i}} \sin\theta = k^{\mathrm{r}} \sin\theta' = k^{\mathrm{t}} \sin\phi \tag{2.125}$$

Equating the first two terms gives the law of reflection, i.e. $\theta = \theta'$. Note that as the incident and reflected wave travel in the same media $k^{\mathrm{i}} = k^{\mathrm{r}}$ so the refractive indices n_1, n_2 of the two media are related to the amplitude of the wave vector through

$$n_1 = \frac{k^{\mathrm{i}} c}{\omega} = \frac{k^{\mathrm{r}} c}{\omega} \tag{2.126}$$

$$n_2 = \frac{k^{\mathrm{t}} c}{\omega} \tag{2.127}$$

Equating the first and third term in Equation 2.125 gives Snell's Law i.e.

$$\frac{k^{\mathrm{i}}}{k^{\mathrm{t}}}\left(\times\frac{c/\omega}{c/\omega}\right) = \frac{n_1}{n_2} = \frac{\sin\phi}{\sin\theta} \tag{2.128}$$

2.7.1 Fresnel Equations

To calculate the reflected and transited amplitudes at a boundary it is convenient to consider two orthogonal cases

- a harmonic wave whose electric field vector is perpendicular to the plane of incidence,
- a harmonic wave whose electric field vector is parallel to the plane of incidence.

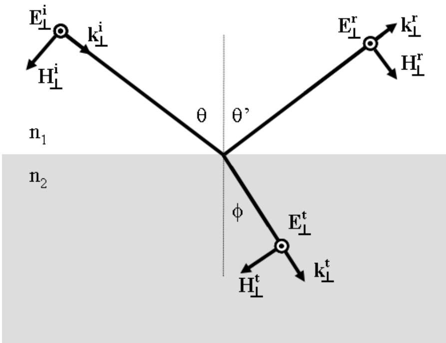


FIGURE 2.18
.

At the boundary the tangential components of the electric and magnetic fields must be continuous. Applying this to the first case shown in Figure 2.18

$$E^{\rm i}_\perp + E^{\rm r}_\perp = E^{\rm t}_\perp \tag{2.129}$$

$$-H^{\rm i}_\perp \cos\theta + H^{\rm r}_\perp \cos\theta' = -H^{\rm t}_\perp \cos\phi \tag{2.130}$$

Using the relationship between the magnitude of the electric and magnetic fields the second equation becomes

$$-k^{\rm i}E^{\rm i}_\perp \cos\theta + k^{\rm r}E^{\rm r}_\perp \cos\theta' = -k^{\rm t}E^{\rm t}_\perp \cos\phi \tag{2.131}$$

Equations 2.129 and 2.131 can be combined to give the ratio of the reflected and transmitted amplitude to the incident amplitude, $r_\perp$ and $t_\perp$ respectively, as

$$r_\perp = \frac{E^{\rm r}_\perp}{E^{\rm i}_\perp} = \frac{n_1\cos\theta - n_2\cos\phi}{n_1\cos\theta + n_2\cos\phi} \tag{2.132}$$

$$t_\perp = \frac{E^{\rm t}_\perp}{E^{\rm i}_\perp} = \frac{2n_1\cos\theta}{(n_1+n_2)\cos\phi} \tag{2.133}$$

which makes use of the fact that $\theta = \theta'$.

In the second case shown in Figure 2.19 the magnetic field is perpendicular to the plane of incidence and the two continuity equations are

$$H^{\rm i}_\parallel + H^{\rm r}_\parallel = H^{\rm t}_\parallel \tag{2.134}$$

$$E^{\rm i}_\parallel \cos\theta - E^{\rm r}_\parallel \cos\theta' = E^{\rm t}_\parallel \cos\phi \tag{2.135}$$

which can be solved to give the ratio of the reflected and transmitted, amplitude to the incident amplitude, $r_\parallel$ and $t_\parallel$ respectively, as

$$r_\parallel = \frac{E^{\rm r}_\parallel}{E^{\rm i}_\parallel} = \frac{n_1\cos\phi - n_2\cos\theta}{n_1\cos\phi + n_2\cos\theta} \tag{2.136}$$

$$t_\parallel = \frac{E^{\rm t}_\parallel}{E^{\rm i}_\parallel} = \frac{2n_1\cos\theta}{n_1\cos\phi + n_2\cos\theta} \tag{2.137}$$

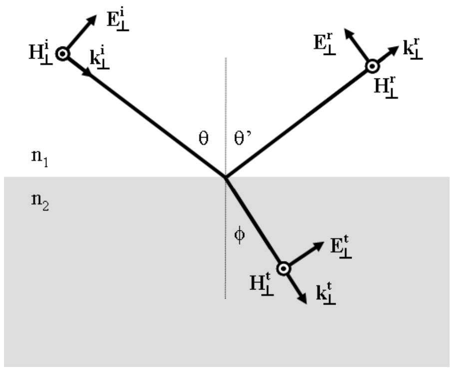


FIGURE 2.19
.

The expressions for the reflected and transmitted amplitudes can be combined with Snell's law and expressed purely in terms of the reflected and refracted angles.

$$r_\perp = -\frac{\sin(\theta - \phi)}{\sin(\theta + \phi)} \tag{2.138}$$

$$t_\perp = \frac{2\cos\theta\sin\phi}{\sin(\theta + \phi)} \tag{2.139}$$

$$r_\parallel = -\frac{\tan(\theta - \phi)}{\tan(\theta + \phi)} \tag{2.140}$$

$$t_\parallel = \frac{2\cos\theta\sin\phi}{\sin(\theta + \phi)\cos(\theta - \phi)} \tag{2.141}$$

These are known as the Fresnel's equations. They amplitude ratios of the reflected light can also be expressed using n_1, n_2 and eliminating ϕ i.e.

$$r_\perp = \frac{\cos\theta - \sqrt{n^2 - \sin^2\theta}}{\cos\theta + \sqrt{n^2 - \sin^2\theta}} \tag{2.142}$$

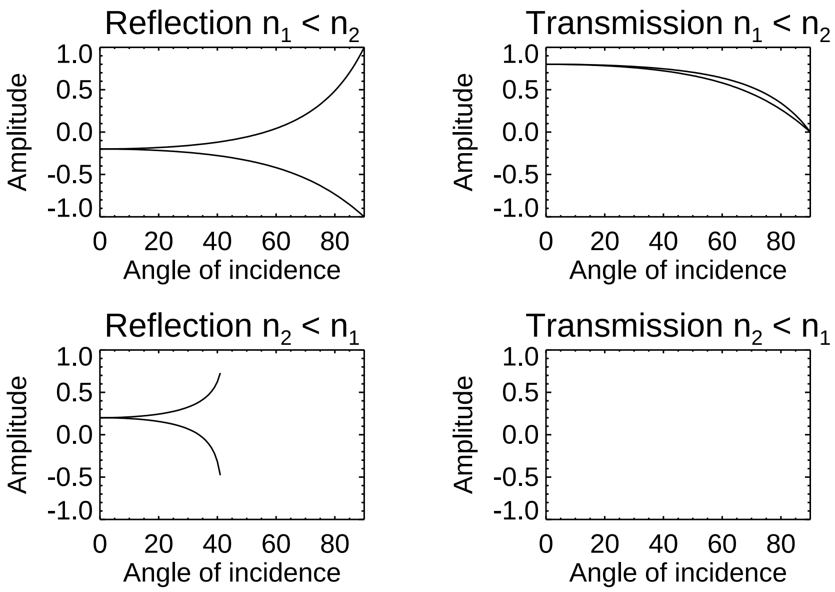
$$r_\parallel = -\frac{-n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}}{n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}} \tag{2.143}$$

where $n = n_2/n_1$.

The reflection and transmission amplitude matrices that encapsulate these processes are expressed

$$\begin{pmatrix} E^{\mathrm{r}}_\parallel \\ E^{\mathrm{r}}_\perp \end{pmatrix} = \begin{pmatrix} r_\parallel & 0 \\ 0 & r_\perp \end{pmatrix} \begin{pmatrix} E^{\mathrm{i}}_\parallel \\ E^{\mathrm{i}}_\perp \end{pmatrix} \tag{2.144}$$

$$\begin{pmatrix} E^{\mathrm{t}}_\parallel \\ E^{\mathrm{t}}_\perp \end{pmatrix} = \begin{pmatrix} t_\parallel & 0 \\ 0 & t_\perp \end{pmatrix} \begin{pmatrix} E^{\mathrm{i}}_\parallel \\ E^{\mathrm{i}}_\perp \end{pmatrix} \tag{2.145}$$



2.7.2 Reflectance and Transmittance

The expressions for the reflected and transmitted amplitudes are just that - further terms are required for the redirection of the energy of the incident field. The energy carried by a ray is proportional to the amplitude squared (ref) so that the energy of the incident wave per unit area at the interface is proportional to $EI^2 \cos\theta$.

The reflectance is defined as the ratio of energy per unit area reflected from the interface to the energy per unit area incident on the interface. The reflectivity perpendicular and parallel to the plane of reflection are then

$$\mathcal{R}_{\parallel} = \frac{\tan^2(\theta - \phi)}{\tan^2(\theta + \phi)}, \tag{2.146}$$

$$\mathcal{R}_{\perp} = \frac{\sin^2(\theta - \phi)}{\sin^2(\theta + \phi)}. \tag{2.147}$$

Similarly the transmittance perpendicular and parallel to the plane of reflection are

$$\mathcal{T}_{\parallel} = \frac{\sin 2\theta \sin 2\phi}{\sin^2(\theta + \phi)\cos^2(\theta - \phi)}, \tag{2.148}$$

$$\mathcal{T}_{\perp} = \frac{\sin 2\theta \sin 2\phi}{\sin^2(\theta + \phi)}. \tag{2.149}$$

Note that the reflectance components are the square of the equivalent reflection amplitudes whereas the transmittance components the square of the equivalent transmission amplitudes multiplied by a $n_2 \cos\phi / n_1 \cos\theta$ term to account for the change in the effective area of the ray as it is refracted. It is possible to verify that

$$\mathcal{R}_{\parallel} + \mathcal{T}_{\parallel} = 1 \tag{2.150}$$

$$\mathcal{R}_{\perp} + \mathcal{T}_{\perp} = 1 \tag{2.151}$$

which is just stating that energy is conserved.

As the two components of the reflected wave have different amplitudes an unpolarized wave may be partly polarized during reflection or transmission. The degree of polarisation, P, is given by

$$P = \left| \frac{\mathcal{R}_{\parallel} - \mathcal{R}_{\perp}}{\mathcal{R}_{\parallel} + \mathcal{R}_{\perp}} \right| \tag{2.152}$$

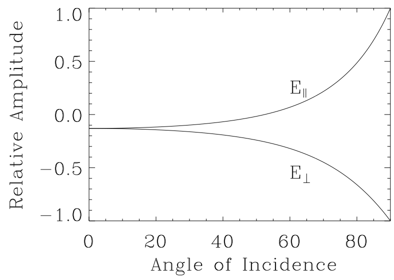


FIGURE 2.20

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The relative amplitudes for the two components and the degree of polarisation are shown in Figure 2.20. A negative ratio equates to a 180 degree phase change during reflection. The Brewster angle occurs when the parallel component is zero so that

$$\theta = \tan^{-1} n \tag{2.153}$$

As the reflect wave contains a single component it has been polarized. For a refractive index of 1.3, typical of water, the Brewster angle occurs at about 52°.

2.8 Scattering

Scattering is the process by which light is redirected by a localised change in the refractive index of a medium. Consider a scatterer placed at the origin illuminated by

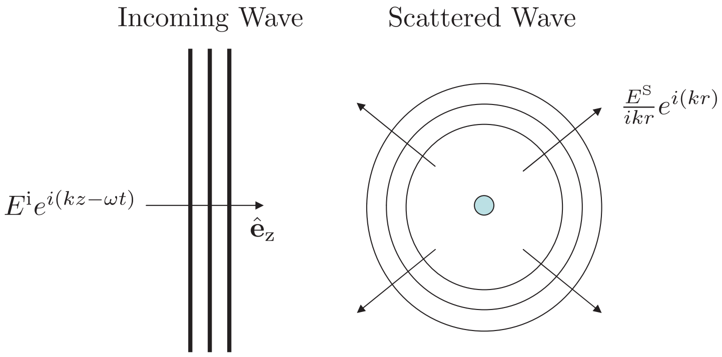

FIGURE 2.21
The scattering of a plane electromagnetic wave.

a plane wave (typicaly travelling in the positive z direction) giving rise to a spherical scattered wave as depicted in Figure 2.21.

The aim of a mathematical description of scattering is to relate the amplitudes of the incident and scattered waves. To achieve this consider a ray that is initially travelling in the z direction before being scattered through an angle Θ as shown in Figure 2.22. The plane containing the direction of propagation of the incident and scattered rays is called the *plane of scattering* and Θ is known as the *scattering angle*. Since any electric vector may be arbitrarily decomposed into orthogonal components we may choose these components perpendicular ($E^s_\perp$) and parallel ($E^s_\|$) to the plane of scattering. The reference plane remains arbitrary for both forward scattering ($\Theta = 0$) and backscattering ($\Theta = 180$). The incident ray is described by its two components of polarization as

$$\mathbf{E}^i_\|(z,t) = \mathbf{E}^i_{\|0} e^{i(kz-\omega t)} \tag{2.154}$$

$$\mathbf{E}^i_\perp(z,t) = \mathbf{E}^i_{\perp 0} e^{i(kz-\omega t)} \tag{2.155}$$

The linearity of Maxwell's equations means that the scattered light will be a linear sum of the two electric field components. The scattered ray is described by its two components of polarization as

$$\mathbf{E}^s_\|(r,t) = \frac{\mathbf{E}^s_{\|0} e^{i(kr-\omega t)}}{ikr} \tag{2.156}$$

$$\mathbf{E}^s_\perp(r,t) = \frac{\mathbf{E}^s_{\perp 0} e^{i(kr-\omega t)}}{ikr} \tag{2.157}$$

where r is the distance from the centre of the scatterer which is located at the origin.

The scattering amplitude matrix , **S** is used to relate the amplitudes of components of the scattered ray to the amplitudes of the components of the incident ray through

$$\begin{pmatrix} \mathbf{E}^s_\| \\ \mathbf{E}^s_\perp \end{pmatrix} = \frac{e^{ikr}}{ikr} \begin{pmatrix} s_2 & s_3 \\ s_4 & s_1 \end{pmatrix} \begin{pmatrix} \mathbf{E}^i_{\|0} \\ \mathbf{E}^i_{\perp 0} \end{pmatrix} e^{i(kz-\omega t)} \tag{2.158}$$

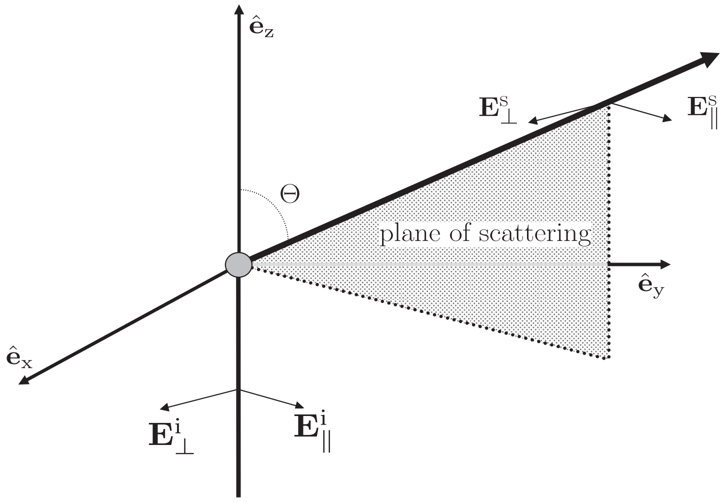

FIGURE 2.22
Geometry of scattering.

The calculation of scattering amplitude matrix values is addressed in Chapter 5 for various types of scatterer.

Problem 2.1 For a linear medium show that the imaginary part of the refractive index can be determine by measuring the dissipation of power in travelling through a known thickness *z*.

Problem 2.2 Rearrange the expressions for E field component amplitudes (Equation 2.101) to obtain Equation 2.102.

Additional Reading

Pedrotti, F. L., L. S. Pedrotti, and L. M. Pedrotti, *Introduction to Optics*, third ed., Pearson Education, Upper Saddle River, 2007