6

Equation of Radiative Transfer

6.1 Microphysical Optical Properties of a Multi-Component Volume

The previous two chapters have discussed the process involved with gaseous absorption, and molecular and particulate scattering respectively. Before developing methods to cope with redirection of radiation on a far larger scale it is important to understand how to combine these process into a single microphysical description of an atmospheric volume. Consider a number of different optical components described by the single scatter albedo, $\tilde{\omega}_i$, the volume extinction coefficient, β_i^{ext} and the phase function $P_i(\omega_0, \omega)$ where the value of the subscript i denotes a particular component. A medium composed of N optical components is equivalent to a homogeneous medium characterised by the following optical properties:

$$\beta^{\text{ext}} = \sum_{i=1}^{N} \beta_i^{\text{ext}} \tag{6.1}$$

$$\tilde{\omega} = \frac{\sum_{i=1}^{N} \tilde{\omega}_i \beta_i^{\text{ext}}}{\sum_{i=1}^{N} \beta_i^{\text{ext}}} \tag{6.2}$$

$$P(\omega_0, \omega) = \frac{\sum_{i=1}^{N} \tilde{\omega}_i \beta_i^{\text{ext}} P_i(\omega_0, \omega)}{\sum_{i=1}^{N} \tilde{\omega}_i \beta_i^{\text{ext}}} \tag{6.3}$$

As an example consider a situation where the known properties are the gaseous absorption represented by $\beta_{\text{gas}}^{\text{ext}} (= \beta_{\text{gas}}^{\text{abs}})$, the phase function for Rayleigh scatter $P_{\text{molecule}}(\omega_0, \omega)$ (the single scatter albedo for Rayleigh scattering is taken as unity) and the single scatter albedo, $\tilde{\omega}_{\text{particle}}$, the volume extinction coefficient, $\beta_{\text{particles}}^{\text{ext}}$ and the phase function $P_{\text{particle}}(\omega_0, \omega)$ for scattering by an ensemble of particles within the volume. The medium may be treated as one whose optical properties are

$$\beta^{\text{ext}} = \beta_{\text{molecule}}^{\text{ext}} + \beta_{\text{particle}}^{\text{ext}} + \beta_{\text{gas}}^{\text{ext}} \tag{6.4}$$

$$\tilde{\omega} = \frac{\beta_{\text{molecule}}^{\text{ext}} + \tilde{\omega}_{\text{particle}} \beta_{\text{particles}}^{\text{ext}}}{\beta_{\text{molecule}}^{\text{ext}} + \beta_{\text{particle}}^{\text{ext}} + \beta_{\text{gas}}^{\text{ext}}} \tag{6.5}$$

$$P(\omega_0, \omega) = \frac{\beta_{\text{molecules}}^{\text{ext}} P_{\text{molecule}}(\omega_0, \omega) + \beta_{\text{particle}}^{\text{ext}} \tilde{\omega}_{\text{particle}} P_{\text{particle}}(\omega_0, \omega)}{\beta_{\text{molecule}}^{\text{ext}} + \beta_{\text{particle}}^{\text{ext}} \tilde{\omega}_{\text{particle}}} \tag{6.6}$$

6.2 Equation of Radiative Transfer

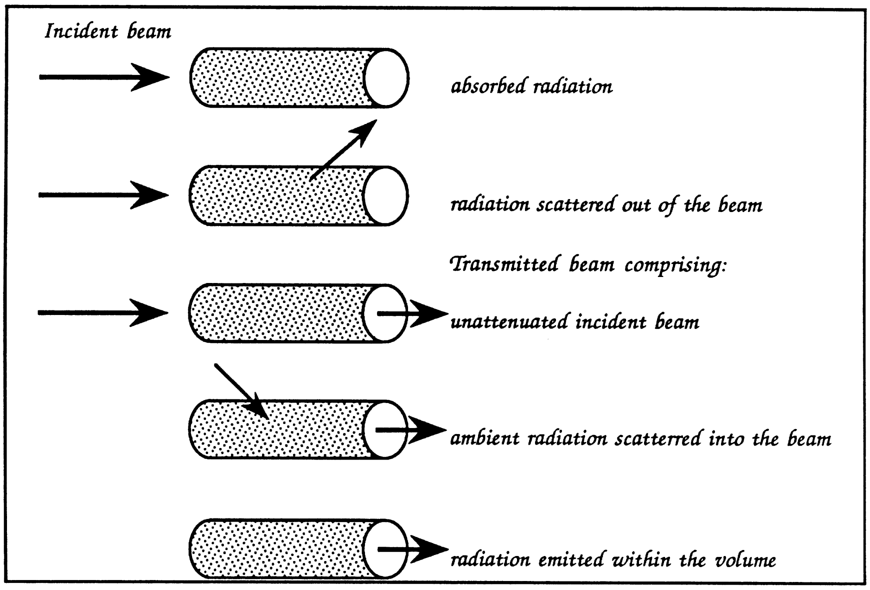

FIGURE 6.1
Elemental radiative transfer processes.

The fundamental equation describing the propagation of electromagnetic radiation is the equation of transfer. Consider an electromagnetic wave travelling through a scattering and absorbing medium in thermal equilibrium with its surroundings. Although the original beam is attenuated by absorption and scattering, it is enhanced by thermal emission and by ambient radiation scattered into the beam (see Figure 6.1). The elemental change in spectral radiance of a beam, $L_\lambda(x, y, z, \boldsymbol{\omega})$, as it transverses a volume element is

$$\begin{aligned}\frac{dL_\lambda(x + dx, y + dy, z + dz, \boldsymbol{\omega})}{dl} = & -\beta^{\text{ext}} L_\lambda(x, y, z, \boldsymbol{\omega}) \\ & +\beta^{\text{sca}} L_\lambda^{\text{S}}(x, y, z, \boldsymbol{\omega}) \\ & +\beta^{\text{abs}} L_\lambda^{\text{B}}(x, y, z, \boldsymbol{\omega}, T). \qquad (6.7)\end{aligned}$$

The three terms on the left hand side of Equation 6.7 denote: the attenuation of the beam, the radiation gained by scattering into direction ($\boldsymbol{\omega}$), and the thermal emission of the volume into the beam ($L_\lambda^{\text{B}}(x, y, z, \boldsymbol{\omega}, T) = B_\lambda(\lambda, T(x, y, z))$). The scattering contribution is found by summing the contributions from the ambient radiation field $L_\lambda^{\text{ambient}}(x, y, z, \boldsymbol{\omega}')$ that is incident on the volume and then scattered into the direction of interest (refer to Equation 5.41), i.e.

$$L_\lambda^{\text{S}}(x, y, z, \boldsymbol{\omega}) = \int_0^{4\pi} \frac{P(\omega', \omega)}{4\pi} L_\lambda^{\text{ambient}}(x, y, z, \boldsymbol{\omega}')\, d\Omega' \tag{6.8}$$

If there are no external sources so then the ambient radiation field is $L_\lambda^{\text{ambient}}(x, y, z, \boldsymbol{\omega}) = L_\lambda(x, y, z, \boldsymbol{\omega})$.

Equation 6.7 can be rewritten using the definition of volume single scatter albedo (Equation 3.140) to give

$$\begin{aligned}\frac{dL_\lambda(x + dx, y + dy, z + dz, \boldsymbol{\omega})}{\beta^{\text{ext}} dl} &= -L_\lambda(x, y, z, \boldsymbol{\omega}) \\ &\quad + \tilde{\omega} L_\lambda^{\text{S}}(x, y, z, \boldsymbol{\omega}) \\ &\quad + (1 - \tilde{\omega}) L_\lambda^{\text{B}}(x, y, z, \boldsymbol{\omega}, T).\end{aligned} \tag{6.9}$$

If we define the **source function** to be the contribution to the beam by scattering and emission, i.e.

$$J_\lambda(x, y, z, \boldsymbol{\omega}, T) = \tilde{\omega} L_\lambda^{\text{S}}(x, y, z, \boldsymbol{\omega}) + (1 - \tilde{\omega}) L_\lambda^{\text{B}}(x, y, z, \boldsymbol{\omega}, T),$$

then we have

$$\frac{dL_\lambda(x + dx, y + dy, z + dz, \boldsymbol{\omega})}{\beta^{\text{ext}} dl} = -L_\lambda(x, y, z, \boldsymbol{\omega}) + J_\lambda(x, y, z, \boldsymbol{\omega}, T). \tag{6.10}$$

This is the general equation of transfer. It is fundamental in the discussion of any radiative transfer process.

Before examining the solutions of the general equation of transfer it is useful to look at two special cases:

1. a medium where there are no scattering or emission sources, and
2. a medium where there are no scattering sources.

6.2.0.1 Equation of Transfer with no Scattering or Emission Sources

If a light beam is travelling through a homogeneous medium where the contribution to the beam by scattering and emission is negligible (i.e. $J_\lambda(x, y, z, \boldsymbol{\omega}, T) = 0$) then Equation 6.10 may be integrated, generating

$$L_\lambda(x', y', z', \boldsymbol{\omega}) = L_\lambda(x, y, z, \boldsymbol{\omega}) e^{-\beta^{\text{ext}} l}, \tag{6.11}$$

where $l = \sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}$. Equation 6.11 is referred to as Bouguer's law.* It is important to note Bouguer's law does not take into account any forward-scattered radiance, and that strictly it only applies to collimated beams. Practically, it is usually applied to beams that are approximately collimated.

*A brief review shows confusion regarding the name of Equation 6.11 e.g. it is called, Beer's law [*Paltridge and Platt*, 1976], Lambert's absorption law [*Slater*, 1980], the Beer-Bouguer-Lambert law [*Liou*,

6.2.0.2 Equation of Transfer with no Scattering Sources

Consider a non-scattering medium which acts as a blackbody and is in local thermodynamic equilibrium. A beam of radiance, L_λ, passing through it will undergo absorption while emission from the matter also takes place. The source function in this case is given by the Planck function

$$J_\lambda(l, \lambda, T) = B_\lambda(\lambda, T(l)). \tag{6.12}$$

where l denotes the position along the path the beam takes through the medium. In this case the equation of transfer may be written as

$$dL_\lambda = -L_\lambda \beta^{\text{ext}} dl + (1 - \tilde{\omega}) B_\lambda(\lambda, T(l)) \beta^{\text{ext}} dl. \tag{6.13}$$

As there is no scattering $\tilde{\omega} = 0$ and β^{ext} $(= \beta^{\text{abs}} + \beta^{\text{sca}}) = \beta^{\text{abs}}$. Hence

$$dL_\lambda = -L_\lambda \beta^{\text{abs}} dl + B_\lambda(T(l)) \beta^{\text{abs}} dl. \tag{6.14}$$

This is Schwartzchild's equation. The first term on the right hand side denotes the reduction in radiance due to absorption whereas the second term represents the increase in radiance arising from blackbody emission.

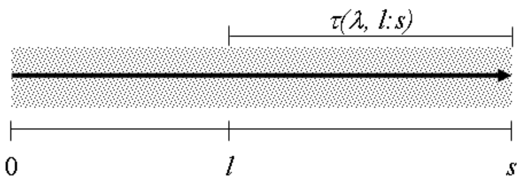

FIGURE 6.2
Emission along a path. The transmittance from a point l along the path to its end is given by $e^{-\tau(\lambda, l:s)}$.

Schwartzchild's equation can be used to calculate the radiance at the end of a path of length s as shown in Figure 6.2. The temperature of the medium a point l, along the path is defined as $T(l)$. The radiance is found by rearranging Equation 6.14 to give

$$L_\lambda + \frac{dL_\lambda}{\beta^{\text{abs}} dl} = B_\lambda(\lambda, T(l)).$$

1980], the Lambert-Bouguer law of transmission [*Chahine*, 1983] and the Bouguer extinction law [*Fenn et al.*, 1985]. In an excellent historical note [*Iqbal*, 1983] points out the relationship was derived experimentally by Piere Bouguer in 1729 and theoretically by Johann Heinrich Lambert in 1760. Beer's law, produced a century later, states that the absorption of radiation depends only on the concentration of the absorbing species and is therefore a restricted form of Bouguer's law.

This expression is multiplied by the transmittance to the end of the path $e^{-\tau(\lambda,l:s)}$ to get

$$L_\lambda \beta^{\text{abs}} e^{-\tau_p(\lambda,l:s)} + \frac{dL_\lambda}{dl} e^{-\tau_p(\lambda,l:s)} = \beta^{\text{abs}} B_\lambda(\lambda, T(l)) e^{-\tau_p(\lambda,l:s)},$$

which when integrated from 0 to s results in

$$\int_0^s \left[L_\lambda \beta^{\text{abs}} e^{-\tau_p(\lambda,l:s)} + \frac{dL_\lambda}{dl} e^{-\tau_p(\lambda,l:s)} \right] dl = \int_0^s \beta^{\text{abs}} B_\lambda(\lambda, T(l)) e^{-\tau_p(\lambda,l:s)}\, dl.$$

Note that if

$$f(x) = L_\lambda e^{-\tau_p(\lambda,l:s)}$$

then

$$\frac{\partial f(x)}{\partial x} = L_\lambda \beta^{\text{abs}} e^{-\tau_p(\lambda,l:s)} + \frac{dL_\lambda}{dl} e^{-\tau_p(\lambda,l:s)}$$

Hence

$$\left[L_\lambda e^{-\tau_p(\lambda,l:s)} \right]_0^s = \int_0^s \beta^{\text{abs}} B_\lambda(\lambda, T(l)) e^{-\tau_p(\lambda,l:s)}\, dl,$$

$$\Rightarrow \left[L_\lambda(s) e^{-\tau_p(\lambda,s:s)} - L_\lambda(0) e^{-\tau_p(\lambda,0:s)} \right] = \int_0^s \beta^{\text{abs}} B_\lambda(\lambda, T(l)) e^{-\tau_p(\lambda,l:s)}\, dl,$$

which leads to the solution

$$L_\lambda(l) = L_\lambda(0) e^{-\tau_p(\lambda,0:s)} + \int_0^s \beta^{\text{abs}} B_\lambda(\lambda, T(l)) e^{-\tau_p(\lambda,l:s)}\, dl. \tag{6.15}$$

The first term represents the radiance that is not attenuation during its passage through the medium from 0 to s. The second term represents the emission contribution from the medium along the path from 0 to s.

6.3 The Plane Parallel Approximation

In a stratified atmosphere atmospheric properties vary sharply with height. It is therefore convenient to treat radiative transfer within the framework of a **plane parallel atmosphere** in which

- Curvature associated with sphericity of the Earth is ignored.
- The medium is regarded as horizontally homogeneous and the radiation field horizontally isotropic.

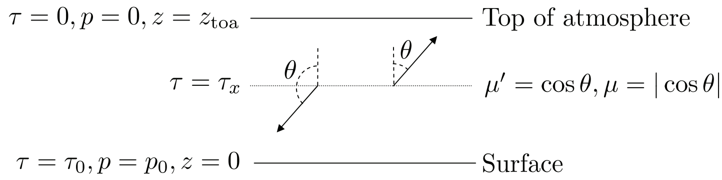

FIGURE 6.3
Plane-parallel coordinates.

Atmospheric light paths are defined by μ the cosine of the zenith angle. The position of radiative quantities can then be reduced from (x, y, z) to (z).

In problems of radiative transfer in plane-parallel atmospheres it is convenient to measure distance vertically in units of **normal monochromatic optical thickness or depth** defined relative to the top of the atmosphere as

$$\tau(z) = \int_{z}^{z_{\text{TOA}}} \beta^{\text{ext}}(z')\, dz'. \tag{6.16}$$

From this definition, τ decreases with height so that $\tau = 0$ at the top of the atmosphere. Hence atmospheric path and optical depth are related by [redacted] [redacted] negative depends upon path direction

$$d\tau = -\beta^{\text{ext}} \cos(\theta)\, dl, \tag{6.17}$$

The $\cos(\theta)$ term accounts for atmospheric slant paths. For a real atmosphere, the slant path includes refraction by the atmosphere as well as spherical geometry. The relative airmass is the ratio of the amount of air in the line of sight (for some zenith angle) to the amount in the zenith. The applicability of the plane parallel approximation can be assessed by comparing the relative airmass factor for a spherical shell to that for a plane-parallel layer. Figure 6.4 show the relative behaviour where the spherical layer is 10 km thick with a 6370 km radius of curvature. The curves agree to better than 1 % for zenith angles up to 75 °. Hence he curvature of the Earth's atmosphere need only be considered for very high Solar or observation zenith angle.

Equation 6.10 can be rewritten

$$\mu' \frac{dL_\lambda(\tau, \boldsymbol{\omega})}{d\tau} = L_\lambda(\tau, \boldsymbol{\omega}) - \tilde{\omega} L^{\text{S}}_\lambda(\tau, \boldsymbol{\omega}) - (1 - \tilde{\omega}) L^{\text{B}}_\lambda(\lambda, T), \tag{6.18}$$

where $\mu' = \cos\theta$. Note that both $\tilde{\omega}$ and T potentially vary within the atmosphere i.e. are functions of τ . Alternatively if the source function is used then the transfer equation can be written

$$\mu' \frac{dL_\lambda(\tau, \boldsymbol{\omega})}{d\tau} = L_\lambda(\tau, \boldsymbol{\omega}) - J_\lambda(\tau, \boldsymbol{\omega}, T). \tag{6.19}$$

One point of confusion encountered using plane parallel coordinates is the sign of μ'. As θ is defined with respect to the normal pointing outward from the atmosphere

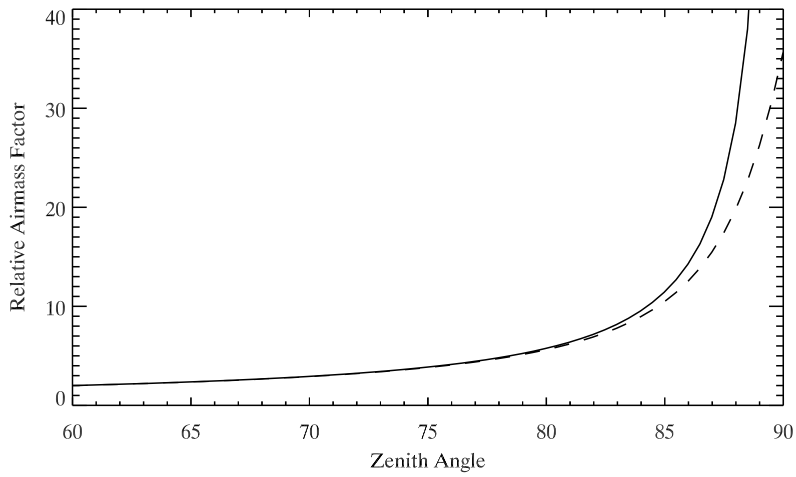

FIGURE 6.4
The relative air ass factor for a plane-parallel atmosphere (solid line) and a curved layer 10 km thick (dashed line).

the value of $\mu' = \cos\theta$ is positive for an upward travelling ray and negative for a downward travelling ray. In this notation the direct transmittance of layer of optical thickness, τ is $e^{-\tau/\mu'}$ for the upward travelling ray and $e^{\tau/\mu'}$ for the downward travelling ray. The irradiance from the Sun (travelling in the downward direction $\boldsymbol{\omega}_0$) onto a surface is $-\mu_0' E^{\text{Sun}}(\boldsymbol{\omega}_0)$. The reversal of sign within the exponential as the ray reverse direction and having to explicitly negate the expression for solar irradiance are avoided if μ is defined as $|\cos\theta|$ and, if appropriate, a negative sign is explicitly shown. The notation adopted is summarised in Table 6.1. Using this notation Equation 6.19 is restated as

$$\mu \frac{dL_\lambda^{\uparrow}(\tau,\boldsymbol{\omega})}{d\tau} = L_\lambda^{\uparrow}(\tau,\boldsymbol{\omega}) - J_\lambda(\tau,\boldsymbol{\omega},T) \qquad 0 < \theta < \pi/2 \qquad (6.20)$$

$$-\mu \frac{dL_\lambda^{\downarrow}(\tau,\boldsymbol{\omega})}{d\tau} = L_\lambda^{\downarrow}(\tau,\boldsymbol{\omega}) - J_\lambda(\tau,\boldsymbol{\omega},T) \qquad \pi/2 < \theta < \pi \qquad (6.21)$$

where the upward and downward radiance fields have been made explicit as $L_\lambda^{\uparrow}(\tau,\boldsymbol{\omega})$ and $L_\lambda^{\downarrow}(\tau,\boldsymbol{\omega})$ respectively.

TABLE 6.1
Definition of angular symbols used to describe the propagation direction of radiation.

Description	Angular Vector	Components	Cosine of the Zenith Angle	Magnitude of Cos(Zenith Angle)
Direction of interest	$\boldsymbol{\omega}$	θ, ϕ	$\mu' = \cos\theta$	$\mu = \lvert\cos\theta\rvert$
Solar direction	$\boldsymbol{\omega}_0$	θ_0, ϕ_0	$\mu'_0 = \cos\theta_0$	$\mu_0 = \lvert\cos\theta_0\rvert$
Any direction	$\boldsymbol{\omega}_i$	θ_i, ϕ_i	$\mu'_i = \cos\theta_i$	$\mu_i = \lvert\cos\theta_i\rvert$

6.3.1 Radiation in a Plane Parallel Atmosphere

To obtain the upward radiance at level τ_x of a finite atmosphere of thickness, τ_0, we multiply Equation 6.20 by $e^{-(\tau-\tau_x)/\mu}$ and integrate it from τ_x to τ_0, i.e.

$$-\frac{1}{\mu}L_\lambda^\uparrow(\tau,\boldsymbol{\omega}) + \frac{dL_\lambda^\uparrow(\tau,\boldsymbol{\omega})}{d\tau} = -\frac{1}{\mu}J_\lambda(\tau,\boldsymbol{\omega},T)$$

$$\int_{\tau_x}^{\tau_0} -\frac{1}{\mu}L_\lambda^\uparrow(\tau,\boldsymbol{\omega})e^{-(\tau-\tau_x)/\mu} + \frac{dL_\lambda^\uparrow(\tau,\boldsymbol{\omega})}{d\tau}e^{-(\tau-\tau_x)/\mu}\,d\tau = \int_{\tau_x}^{\tau_0} -\frac{1}{\mu}J_\lambda(\tau,\boldsymbol{\omega},T)e^{-(\tau-\tau_x)/\mu}\,d\tau$$

$$\left[L_\lambda^\uparrow(\tau,\boldsymbol{\omega})e^{-(\tau-\tau_x)/\mu}\right]_{\tau_x}^{\tau_0} = -\frac{1}{\mu}\int_{\tau_x}^{\tau_0} J_\lambda(\tau,\boldsymbol{\omega},T)e^{-(\tau-\tau_x)/\mu}\,d\tau$$

Hence

$$L_\lambda^\uparrow(\tau_x,\boldsymbol{\omega}) = L_\lambda^\uparrow(\tau_0,\boldsymbol{\omega})e^{-(\tau_0-\tau_x)/\mu} + \frac{1}{\mu}\int_{\tau_x}^{\tau_0} J_\lambda(\tau,\boldsymbol{\omega},T)e^{-(\tau-\tau_x)/\mu}\,d\tau \tag{6.22}$$

To get the downward radiance at level τ_x we multiply Equation 6.21 by $e^{-(\tau_x-\tau)/\mu}$ and integrate from 0 to τ_x, i.e.

$$\frac{1}{\mu}L_\lambda^\downarrow(\tau,\boldsymbol{\omega}) + \frac{dL_\lambda^\downarrow(\tau,\boldsymbol{\omega})}{d\tau} = \frac{1}{\mu}J_\lambda(\tau,\boldsymbol{\omega},T)$$

$$\int_0^{\tau_x} \frac{1}{\mu}L_\lambda^\downarrow(\tau,\boldsymbol{\omega})e^{-(\tau_x-\tau)/\mu} + \frac{dL_\lambda^\downarrow(\tau,\boldsymbol{\omega})}{d\tau}e^{-(\tau_x-\tau)/\mu}\,d\tau = \int_0^{\tau_x} \frac{1}{\mu}J_\lambda(\tau,\boldsymbol{\omega},T)e^{-(\tau_x-\tau)/\mu}\,d\tau$$

$$\left[L_\lambda^\downarrow(\tau,\boldsymbol{\omega})e^{-(\tau_x-\tau)/\mu}\right]_0^{\tau_x} = \frac{1}{\mu}\int_0^{\tau_x} J_\lambda(\tau,\boldsymbol{\omega},T)e^{-(\tau_x-\tau)/\mu}\,d\tau$$

$$L_\lambda^\downarrow(\tau_x,\boldsymbol{\omega}) = L_\lambda^\downarrow(0,\boldsymbol{\omega})e^{-\tau_x/\mu} + \frac{1}{\mu}\int_0^{\tau_x} J_\lambda(\tau,\boldsymbol{\omega},T)e^{-(\tau_x-\tau)/\mu}\,d\tau \tag{6.23}$$

The two special cases of these equations which are most often used are: calculating the upward radiance at the top of the atmosphere and calculating the emergent downward radiance at the bottom of the atmosphere, i.e.

$$L_\lambda^\uparrow(0,\boldsymbol{\omega}) = L_\lambda^\uparrow(\tau_0,\boldsymbol{\omega})e^{-\tau_0/\mu} + \frac{1}{\mu}\int_0^{\tau_0} J_\lambda(\tau,\boldsymbol{\omega},T)e^{-\tau/\mu}\,d\tau \tag{6.24}$$

and

$$L_\lambda^{\downarrow}(\tau_0, \boldsymbol{\omega}) = L_\lambda^{\downarrow}(0, \boldsymbol{\omega}) e^{-\tau_0/\mu} + \frac{1}{\mu} \int_0^{\tau_0} J_\lambda(\tau, \boldsymbol{\omega}, T) e^{-(\tau_0 - \tau)/\mu} \, d\tau \tag{6.25}$$

6.3.2 Components of the Source Function

To evaluate these expressions the value of the source function that generates the radiation field must be known. For the Earth's atmosphere three components of the source function are considered: the diffuse field sources, $J_\lambda^{\text{diffuse}}(\tau, \boldsymbol{\omega})$, the solar source $J_\lambda^{\text{solar}}(\tau, \boldsymbol{\omega})$, and the internal thermal sources, $J_\lambda^{\text{thermal}}(\tau, \boldsymbol{\omega})$. Using these terms the source function is expressed as

$$J_\lambda(\tau, \boldsymbol{\omega}, T) = J_\lambda^{\text{diffuse}}(\tau, \boldsymbol{\omega}) + J_\lambda^{\text{solar}}(\tau, \boldsymbol{\omega}) + J_\lambda^{\text{thermal}}(\tau, \boldsymbol{\omega}, T) \tag{6.26}$$

where the individual terms are defined:

$$J_\lambda^{\text{diffuse}}(\tau, \boldsymbol{\omega}) = \tilde{\omega} \int_0^{4\pi} L_\lambda(\tau, \boldsymbol{\omega}_i) \frac{P(\boldsymbol{\omega}_i, \boldsymbol{\omega})}{4\pi} \, d\boldsymbol{\omega}_i \tag{6.27}$$

$$J_\lambda^{\text{solar}}(\tau, \boldsymbol{\omega}) = \frac{\tilde{\omega} E_\lambda^0(\omega_0) P(\tau, \omega_0, \omega) e^{-\tau/\mu_0}}{4\pi} \tag{6.28}$$

$$J_\lambda^{\text{thermal}}(\tau, \boldsymbol{\omega}, T) = (1 - \tilde{\omega}) B_\lambda(\lambda, T(\tau)). \tag{6.29}$$

Although the equation of radiative transfer has been developed including expressions for both scattering and emission processes it is generally not necessary to consider all radiation sources. Instead the spectrum is dived into three regimes: for wavelengths less than about $3.5\,\mu$m um the atmosphere is illuminated by the Sun and thermal sources are neglected, for wavelengths longer than about $8.9\,\mu$m the contribution to the radiance field by sunlight can be ignored. In the intervening region both terrestrial and solar sources may need to be considered.

6.3.3 A Plane-Parallel Atmosphere with a Solar Source

If the atmosphere is illuminated with irradiance $E_\lambda^0(\boldsymbol{\omega}_0)$ from a single directional source, i.e. the Sun, then the radiation field is composed of the diffuse field created by scattering within the volume and the imposed field caused by the external illumination, i.e.

$$E_\lambda^{\text{ambient}}(\tau, \boldsymbol{\omega}) = L_\lambda^{\text{diffuse}}(\tau, \boldsymbol{\omega}) d\boldsymbol{\omega} + E_\lambda^0(\boldsymbol{\omega}_0) \delta_{\phi,\phi_0} \delta_{\theta,\theta_0} e^{-\tau/\cos\theta_0} \tag{6.30}$$

The radiance field is found from a source function that combines the scattered and solar fields i.e.

$$J_\lambda^{\text{solar}}(\tau, \boldsymbol{\omega}) = \tilde{\omega} \int_0^{4\pi} L_\lambda(\tau, \boldsymbol{\omega}_i) \frac{P(\boldsymbol{\omega}_i, \boldsymbol{\omega})}{4\pi} \, d\boldsymbol{\omega}_i + \frac{\tilde{\omega} E_\lambda^0(\boldsymbol{\omega}_0) P(\tau, \boldsymbol{\omega}_0, \boldsymbol{\omega}) e^{-\tau/\mu_0}}{4\pi} \tag{6.31}$$

Substituting this into Equation 6.22 gives the upward radiance field at optical depth τ_x in a layer of thickness τ_0 i.e.

$$L_\lambda^\uparrow(\tau_x,\boldsymbol{\omega}) = L_\lambda^\uparrow(\tau,\boldsymbol{\omega})e^{-(\tau_0-\tau_x)/\mu} + \frac{\tilde{\omega}}{\mu}\int_{\tau_x}^{\tau_0}\int_0^{4\pi} L_\lambda(\tau,\boldsymbol{\omega}_i)\frac{P(\boldsymbol{\omega}_i,\boldsymbol{\omega})}{4\pi}\,d\boldsymbol{\omega}_i e^{-(\tau-\tau_x)/\mu}\,d\tau$$
$$+\frac{\tilde{\omega}}{\mu}\int_{\tau_x}^{\tau_0}\frac{E_\lambda^0(\boldsymbol{\omega}_0)P(\tau,\boldsymbol{\omega}_0,\boldsymbol{\omega})e^{-\tau/\mu_0}}{4\pi}e^{-(\tau-\tau_x)/\mu}\,d\tau \quad (6.32)$$

The corresponding downward field (from substitution of the source function into Equation 6.23) is

$$L_\lambda^\downarrow(\tau_x,\boldsymbol{\omega}) = L_\lambda^\downarrow(0,\boldsymbol{\omega})e^{-\tau_x/\mu} + \frac{\tilde{\omega}}{\mu}\int_0^{\tau_x}\int_0^{4\pi} L_\lambda(\tau,\boldsymbol{\omega}_i)\frac{P(\boldsymbol{\omega}_i,\boldsymbol{\omega})}{4\pi}\,d\boldsymbol{\omega}_i e^{-(\tau_x-\tau)/\mu}\,d\tau$$
$$+\frac{\tilde{\omega}}{\mu}\int_0^{\tau_x}\frac{E_\lambda^0(\boldsymbol{\omega}_0)P(\tau,\boldsymbol{\omega}_0,\boldsymbol{\omega})e^{-\tau/\mu_0}}{4\pi}e^{-(\tau_x-\tau)/\mu}\,d\tau \quad (6.33)$$

6.3.4 A Plane-Parallel Atmosphere with Thermal Sources

Consider a non-scattering atmosphere of optical depth τ_0 above a black surface at temperature T_0 where there are no extraterrestrial sources so the source function is given by Equation 6.29. Then from Equations 6.22 and 6.23 the upward and downward radiances in direction $\boldsymbol{\omega}$ are

$$L_{\tilde{\nu}}^\uparrow(\tau_x,\boldsymbol{\omega}) = B_{\tilde{\nu}}(\tilde{\nu},T_0)e^{-(\tau_0-\tau_x)/\mu} + \frac{1}{\mu}\int_{\tau_x}^{\tau_0}(1-\tilde{\omega})B_{\tilde{\nu}}(\tilde{\nu},T(\tau))e^{-(\tau-\tau_x)/\mu}\,d\tau \quad (6.34)$$

$$L_{\tilde{\nu}}^\downarrow(\tau_x,\boldsymbol{\omega}) = \frac{1}{\mu}\int_0^{\tau_x}(1-\tilde{\omega})B_{\tilde{\nu}}(\tilde{\nu},T(\tau))e^{-(\tau_x-\tau)/\mu}\,d\tau \quad (6.35)$$

The $1-\tilde{\omega}$ term has been left inside the integrals as this term will normally vary with optical depth.

6.4 Expansion of the Radiance Field as a Fourier Series in Azimuth

The radiance, $L_\lambda(\tau,\boldsymbol{\omega})$, in a plane parallel atmosphere can be expressed as a Fourier cosine series i.e.

$$L_\lambda(\tau,\boldsymbol{\omega}) = \sum_{m=0}^{\infty} L_\lambda^m(\tau,\mu)\cos m(\phi-\phi_0) \quad (6.36)$$

where ϕ_0 is an arbitrary reference azimuth direction. The coefficients, $L_\lambda^m(\tau,\mu)$, are defined by

$$L_\lambda^m(\tau,\mu) = (2-\delta_{0,m})\frac{1}{2\pi}\int_0^{2\pi} L_\lambda(\tau,\boldsymbol{\omega})\cos m(\phi-\phi_0)\,d\phi. \quad (6.37)$$

From this expression it can be seen that the first expansion term in Equation 6.36 is the mean field and that the additional terms represent symmetric perturbations about the mean. As symmetric changes in radiance with azimuth do not contribute to the irradiance (see Problem 6.3)

$$E^{\uparrow} = 2\pi \int_0^1 L_\lambda^0(\tau, \mu)\mu \, d\mu. \tag{6.38}$$

Substituting Fourier expansion of the radiance field into Equation 6.19 gives independent equations of the form

$$\mu \frac{dL_\lambda^m(\tau, \mu)}{d\tau} = L_\lambda^m(\tau, \mu) - J_\lambda^m(\tau, \mu, T). \tag{6.39}$$

where $J_\lambda^m(\tau, \mu T)$ is the m^{th} Fourier component of the source function. Each of these equations is solved for $L_\lambda^m(\tau, \mu)$ and the radiance field reconstructed using Equation 6.36. The computational advantage comes from the fact the actual number of Fourier terms, N, needed to capture the azimuthal variation is smaller than the number of quadrature angles in azimuth required to achieve the same accuracy.

The number of Fourier expansion terms depends strongly on μ_i and μ_r, as observed by *Dave and Gazdag* [1970] and *Hansen and Pollack* [1970]. The reflection is azimuthally independent when μ_i or $\mu_r = 1$. In this situation the single term $n = 0$ fully describes the angular scattering pattern. The highest number of terms is required for the case $\mu_i = \mu_r = 0$. Empirical results presented by *van de Hulst* [1980] give the number of terms $N = 25 \sin\theta$, where θ is the minimum of θ_r and θ_i. *King* [1983] concludes that about l6 terms is adequate for most remote sensing applications.

6.4.1 Expansion of the Source Function Components

In order to evaluate the set of equations defined in Equation 6.39 the source function must also be decomposed into a cosine series in azimuth. Previously the input and output directions to a medium have been described using two sets of coordinate pairs e.g. $\boldsymbol{\omega}_i = (\theta_i, \phi_i)$ and $\boldsymbol{\omega} = (\theta, \phi)$. A more helpful notation for a Fourier expansion representation of the radiance field is to use the cosines of the input and output zenith angles, denoted by μ_i and μ respectively, and the output azimuth direction as the reference azimuth direction.

In terms of spherical polar coordinates (of unit magnitude) the incident $\vec{x}_i$ and scattered $\vec{x}_s$ directions are expressed

$$\vec{x}_i = \sin\theta_i \cos\theta_i \vec{\imath} + \sin\theta_i \sin\phi_i \vec{\jmath} + \cos\theta_i \vec{k} \tag{6.40}$$

$$\vec{x} = \sin\theta \cos\theta \vec{\imath} + \sin\theta \sin\phi \vec{\jmath} + \cos\theta \vec{k} \tag{6.41}$$

where $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$ are the unit vectors in the x,y and z directions respectively. Figure 6.5 shows the relevant geometry. The scattering angle is found from the dot product of

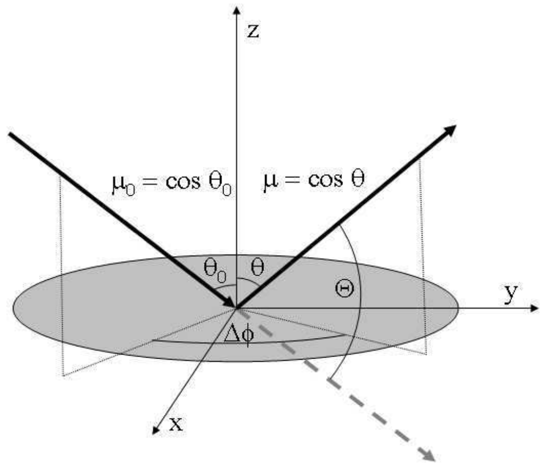

FIGURE 6.5
Relation between angles in the scattering place.

these two vectors i.e.

$$\cos\Theta = \sin\theta_i\cos\phi_i\sin\theta_i\cos\phi_i + \sin\theta_i\sin\phi_i\sin\theta\sin\phi + \cos\theta_i\cos\theta \quad (6.42)$$

$$= \cos\theta_i\cos\theta + \sin\theta_i\sin\theta\cos(\phi_i - \phi) \quad (6.43)$$

$$= \mu_i\mu + \sqrt{1-\mu_i^2}\sqrt{1-\mu^2}\cos(\phi_i - \phi) \quad (6.44)$$

Following *Liou* [1980] the Legendre expansion of the phase function (see Section 3.9.2) can be expressed as

$$P(\mu_i, \mu, \phi_i, \phi) = \sum_{m=0}^{N}(2-\delta_{0,m})\sum_{l=m}^{N}\omega_l\frac{(l-m)!}{(l+m)!}P_l^m(\mu_i)P_l^m(\mu)\cos m(\phi_i - \phi) \quad (6.45)$$

where $P_l^m(\mu)$ is an associated Legendre polynomial and ω_l is the Legendre expansion coefficient of the phase function. The Fourier components of the source function expansions can be found as follows:

Diffuse Substituting Equation 6.45 into the expression for the diffuse source function gives

$$J_\lambda^{\text{diffuse}}(\tau, \omega) = \frac{\tilde{\omega}}{4\pi}\int_0^{2\pi}\int_{-1}^{1}L_\lambda(\tau, \mu_i, \phi_i) \sum_{m=0}^{N}(2-\delta_{0,m})\sum_{l=m}^{N}\omega_l\frac{(l-m)!}{(l+m)!}P_l^m(\mu_i)P_l^m(\mu)\cos m(\phi_i - \phi)\,d\mu_i\,d\phi_i \quad (6.46)$$

which can be rearranged to

$$J_\lambda^{\text{diffuse}}(\tau,\boldsymbol{\omega}) = \sum_{m=0}^{N}\sum_{l=m}^{N}\frac{\tilde{\omega}}{4\pi}\int_{-1}^{1}\omega_l\frac{(l-m)!}{(l+m)!}$$
$$P_l^m(\mu_i)P_l^m(\mu)(2-\delta_{0,m})\int_0^{2\pi}L_\lambda(\tau,\mu_i,\phi_i)\cos m(\phi_i-\phi)\,d\phi_i\,d\mu_i \tag{6.47}$$

Completing the integral over ϕ gives

$$J_\lambda^{\text{diffuse}}(\tau,\boldsymbol{\omega}) = \sum_{m=0}^{N}\sum_{l=m}^{N}\frac{\tilde{\omega}}{4\pi}\int_{-1}^{1}\omega_l\frac{(l-m)!}{(l+m)!}P_l^m(\mu_i)P_l^m(\mu)L_\lambda^m(\tau,\mu_i)\,d\mu_i \tag{6.48}$$

This function can be compared with the Fourier expansion where the reference azimuth angle ϕ_0 has been chosen to be ϕ i.e.

$$J_\lambda^{\text{diffuse}}(\tau,\mu) = \sum_{m=0}^{\infty}J_\lambda^m(\tau,\mu,T)\cos m(\phi-\phi_0) = \sum_{m=0}^{\infty}J_\lambda^m(\tau,\mu,T). \tag{6.49}$$

Then the m^{th} term of the diffuse source function is

$$J_\lambda^{\text{diffuse},m}(\tau,\mu) = \frac{\tilde{\omega}}{4\pi}\sum_{l=m}^{N}\int_{-1}^{1}\omega_l\frac{(l-m)!}{(l+m)!}P_l^m(\mu_i)P_l^m(\mu)L_\lambda^m(\tau,\mu_i)\,d\mu_i \tag{6.50}$$

which can also be obtained from the expression for the Fourier coefficients applied to the source function (see Problem 6.2).

Solar Substituting the expansion of the phase function, Equation 6.45, into the expression for the Solar source function (Equation 6.28) gives

$$J_\lambda^{\text{solar}}(\tau,\boldsymbol{\omega}) = \frac{\tilde{\omega}}{4\pi}E_\lambda^0(\boldsymbol{\omega}_0)\sum_{m=0}^{N}\sum_{l=m}^{N}\omega_l(2-\delta_{0,m})\frac{(l-m)!}{(l+m)!}P_l^m(\mu_0)P_l^m(\mu)\cos m\Delta\phi e^{-\tau/\cos\theta_0} \tag{6.51}$$

which can be expressed as

$$J_\lambda^{\text{solar}}(\tau,\boldsymbol{\omega}) = \sum_{m=0}^{N}(2-\delta_{0,m})\frac{\tilde{\omega}}{4\pi}E_\lambda^0(\boldsymbol{\omega}_0)e^{-\tau/\mu_0}\sum_{l=m}^{N}\omega_l\frac{(l-m)!}{(l+m)!}P_l^m(\mu_0)P_l^m(\mu)\cos m\Delta\phi \tag{6.52}$$

which has the same form as a Fourier cosine expansion in azimuth where the m^{th} Fourier coefficient is

$$J_\lambda^{\text{solar},m}(\tau,\mu) = (2-\delta_{0,m})\frac{\tilde{\omega}}{4\pi}E_\lambda^0(\boldsymbol{\omega}_0)e^{-\tau/\mu_0}\sum_{l=m}^{N}\omega_l\frac{(l-m)!}{(l+m)!}P_l^m(\mu_0)P_l^m(\mu). \tag{6.53}$$

Thermal The m^{th} Fourier coefficient of the expansion of the thermal source function (from Equation 6.29) is

$$J_\lambda^m(\tau, \mu, T) = (2 - \delta_{0,m}) \frac{1}{2\pi} \int_0^{2\pi} (1 - \tilde{\omega}) B_\lambda(\lambda, T(\tau)) \cos m\Delta\phi_0 \, d\phi. \tag{6.54}$$

As the radiance from a black body is isotropic only the first order term is non-zero which gives

$$J_\lambda^m(\tau, \mu, T) = (2 - \delta_{0,m})(1 - \tilde{\omega}) B_\lambda(\lambda, T(\tau)). \tag{6.55}$$

6.5 Approximate Solution Methods

6.5.1 Single Scattering Approximation for a Plane-Parallel Atmosphere

The Equation of Radiative Transfer for an upward propagating ray in a plane parallel atmosphere with no thermal sources is written

$$\mu \frac{dL_\nu(\tau; \boldsymbol{\omega})}{d\tau} = L_\nu(\tau; \boldsymbol{\omega}) - L_\nu^{\text{S}}(\tau; \boldsymbol{\omega}). \tag{6.56}$$

where the source function $L_\nu^{\text{S}}(\tau, \boldsymbol{\omega})$ represents the light from the sphere surrounding the scattering volume that is scattered into the direction $\boldsymbol{\omega} = (\mu, \phi)$ and is given by

$$L_\nu^{\text{S}}(\tau; \boldsymbol{\omega}) = \int_0^{4\pi} \tilde{\omega} L(\tau; \boldsymbol{\omega}_i) P(\boldsymbol{\omega}_i, \boldsymbol{\omega}) \, d\boldsymbol{\omega}_i. \tag{6.57}$$

In Equation 6.57 it is being implicitly assumed that the medium is homogeneous so that the scattering phase function $P(\boldsymbol{\omega}_i, \boldsymbol{\omega})$ is independent of optical depth and the orientation of the scattering volume. For optically thin atmospheres the probability of light being scattering more than once is so small that, to a good approximation, multiple scatters may be ignored. The unscattered radiation field then comprises the attenuated solar beam and the source function is

$$L_\nu^{\text{S}}(\tau; \boldsymbol{\omega}) = \tilde{\omega} E^{\text{Sun}} e^{-\tau/\mu_0} \frac{P(\boldsymbol{\omega}_0, \boldsymbol{\omega})}{4\pi} \tag{6.58}$$

where $\boldsymbol{\omega}_0 = (\theta_0, \phi_0)$ is the direction vector for the solar beam. Substituting the expression for the source function into the equation of transfer gives

$$\mu \frac{dL_\nu(\tau; \boldsymbol{\omega})}{d\tau} = L_\nu(\tau; \boldsymbol{\omega}) - \tilde{\omega} E^{\text{Sun}} e^{-\tau/\mu_0} \frac{P(\boldsymbol{\omega}_0, \boldsymbol{\omega})}{4\pi}. \tag{6.59}$$

This equation is multiplied by $e^{-\tau/\mu}/\mu$ to give

$$\frac{dL_\nu(\tau; \boldsymbol{\omega})}{d\tau} e^{-\tau/\mu} = L_\nu(\tau; \boldsymbol{\omega}) \frac{e^{-\tau/\mu}}{\mu} - e^{-\tau/\mu} \tilde{\omega} E^{\text{Sun}} e^{-\tau/\mu_0} \frac{P(\boldsymbol{\omega}_0, \boldsymbol{\omega})}{4\pi\mu} \tag{6.60}$$

which can be expressed as

$$\frac{d}{d\tau}\left[L_\nu(\tau;\boldsymbol{\omega})e^{-\tau/\mu}\right] = -e^{-\tau/\mu}\tilde{\omega}E^{\text{Sun}}e^{-\tau/\mu_0}\frac{P(\boldsymbol{\omega}_0,\boldsymbol{\omega})}{4\pi\mu} \tag{6.61}$$

Integrating from the top of the atmosphere ($\tau = 0$) to the surface ($\tau = \tau^*$) gives the single scattered radiance field at the top of the atmosphere i.e.

$$\int_0^{\tau^*}\frac{d}{d\tau}\left[L_\nu(\tau;\boldsymbol{\omega})e^{-\tau/\mu}\right]d\tau = -\int_0^{\tau^*}e^{-\tau/\mu}\tilde{\omega}E^{\text{Sun}}e^{-\tau/\mu_0}\frac{P(\boldsymbol{\omega}_0,\boldsymbol{\omega})}{4\pi\mu}\,d\tau \tag{6.62}$$

$$L_\nu(\tau^*;\boldsymbol{\omega})e^{-\tau^*/\mu} - L_\nu(0;\boldsymbol{\omega}) = \tilde{\omega}E^{\text{Sun}}\frac{P(\boldsymbol{\omega}_0,\boldsymbol{\omega})}{4\pi\mu}\left[\frac{1}{1/\mu+1/\mu_0}e^{-\tau(1/\mu+1/\mu_0)}\right]_0^{\tau^*} \tag{6.63}$$

$$= \tilde{\omega}E^{\text{Sun}}\frac{P(\boldsymbol{\omega}_0,\boldsymbol{\omega})}{4\pi}\frac{\mu_0}{\mu+\mu_0}\left[e^{-\tau^*(1/\mu+1/\mu_0)} - 1\right] \tag{6.64}$$

If there is no source of radiation at the bottom of the atmosphere in direction $\boldsymbol{\omega}$ then $L_\nu(\tau^*;\boldsymbol{\omega}) = 0$ and the TOA radiation in direction $\boldsymbol{\omega}$ is

$$L_\nu(0;\boldsymbol{\omega}) = \tilde{\omega}E^{\text{Sun}}\frac{P(\boldsymbol{\omega}_0,\boldsymbol{\omega})}{4\pi}\frac{\mu_0}{\mu_0+\mu}\left[1 - e^{-\tau^*(1/\mu+1/\mu_0)}\right] \tag{6.65}$$

For a downward propagating beam the equation of radiative transfer is expressed

$$-\mu\frac{dL_\nu(\tau;\boldsymbol{\omega})}{d\tau} = L_\nu(\tau;\boldsymbol{\omega}) - L_\nu^{\text{S}}(\tau;\boldsymbol{\omega}). \tag{6.66}$$

Substituting in the solar source function gives

$$-\mu\frac{dL_\nu(\tau;\boldsymbol{\omega})}{d\tau} = L_\nu(\tau;\boldsymbol{\omega}) - \tilde{\omega}E^{\text{Sun}}e^{-\tau/\mu_0}\frac{P(\boldsymbol{\omega}_0,\boldsymbol{\omega})}{4\pi} \tag{6.67}$$

which after multiplying through by $e^{\tau/\mu}/\mu$ can be expressed as

$$\frac{d}{d\tau}\left[L_\nu(\tau;\boldsymbol{\omega})e^{\tau/\mu}\right] = e^{\tau/\mu}\tilde{\omega}E^{\text{Sun}}e^{-\tau/\mu_0}\frac{P(\boldsymbol{\omega}_0,\boldsymbol{\omega})}{4\pi\mu} \tag{6.68}$$

Integrating from 0 to τ^* now gives the single scattered radiance field at the bottom of the atmosphere i.e.

$$L_\nu(\tau^*;\boldsymbol{\omega})e^{\tau^*/\mu} - L_\nu(0;\boldsymbol{\omega}) = \int_0^{\tau^*}\tilde{\omega}E^{\text{Sun}}\frac{P(\boldsymbol{\omega}_0,\boldsymbol{\omega})}{4\pi\mu}e^{\tau(1/\mu-1/\mu_0)}\,d\tau \tag{6.69}$$

$$= \tilde{\omega}E^{\text{Sun}}\frac{P(\boldsymbol{\omega}_0,\boldsymbol{\omega})}{4\pi\mu}\left[\frac{1}{1/\mu-1/\mu_0}e^{\tau(1/\mu-1/\mu_0)}\right]_0^{\tau^*} \tag{6.70}$$

$$= \tilde{\omega}E^{\text{Sun}}\frac{P(\boldsymbol{\omega}_0,\boldsymbol{\omega})}{4\pi}\frac{\mu_0}{\mu_0-\mu}\left[e^{\tau^*(1/\mu-1/\mu_0)} - 1\right] \tag{6.71}$$

If there is no source of radiation at the top of the atmosphere in direction $\boldsymbol{\omega}$ then $L_\nu(0;\boldsymbol{\omega}) = 0$ and the BOA radiation in direction $\boldsymbol{\omega}$ when $\mu \neq \mu_0$ is

$$L_\nu(\tau^*;\boldsymbol{\omega}) = \tilde{\omega}E^{\text{Sun}}\frac{P(\boldsymbol{\omega}_0,\boldsymbol{\omega})}{4\pi}\frac{\mu_0}{\mu_0-\mu}\left[e^{\tau^*(1/\mu-1/\mu_0)} - 1\right]e^{-\tau^*/\mu} \tag{6.72}$$

$$= \tilde{\omega}E^{\text{Sun}}\frac{P(\boldsymbol{\omega}_0,\boldsymbol{\omega})}{4\pi}\frac{\mu_0}{\mu_0-\mu}\left[e^{-\tau^*/\mu_0} - e^{-\tau^*/\mu}\right] \tag{6.73}$$

If $\mu = \mu_0$ Equation 6.68 becomes

$$\frac{d}{d\tau}\left[L_\nu(\tau;\boldsymbol{\omega})e^{\tau/\mu}\right] = = \tilde{\omega}E^{\text{Sun}}\frac{P(\boldsymbol{\omega}_0,\boldsymbol{\omega})}{4\pi\mu} \qquad [\mu = \mu_0] \quad (6.74)$$

which can be integrated to give

$$L_\nu(\tau^*;\boldsymbol{\omega})e^{\tau^*/\mu_0} - L_\nu(0;\boldsymbol{\omega}) = \tilde{\omega}E^{\text{Sun}}\frac{P(\boldsymbol{\omega}_0,\boldsymbol{\omega})}{4\pi\mu}\tau^*. \quad [\mu = \mu_0] \quad (6.75)$$

The downward scattered field is then

$$L_\nu(\tau^*;\boldsymbol{\omega}) = \tilde{\omega}E^{\text{Sun}}\frac{P(\boldsymbol{\omega}_0,\boldsymbol{\omega})}{4\pi\mu}\tau^* e^{-\tau^*/\mu}. \qquad [\mu = \mu_0] \quad (6.76)$$

Note that the total radiance in this direction will also include the unscattered solar radiance.

For very thin atmospheres the exponential terms in Equations 6.65, 6.73 and 6.76 can be approximated to first order in τ^* so that both the reflected and transmitted diffuse fields are expressed by

$$\left.\begin{array}{r} L_\nu(0;\boldsymbol{\omega}) \\ L_\nu(\tau^*;\boldsymbol{\omega}) \end{array}\right\} = \tilde{\omega}E^{\text{Sun}}\frac{P(\boldsymbol{\omega}_0,\boldsymbol{\omega})}{4\pi\mu}\tau^* \qquad [\tau^* \ll 1] \quad (6.77)$$

Equation 6.77 is very helpful in understanding the structure of the diffuse radiation field for thin atmospheres:

- the diffuse radiance is proportional to the optical thickness modulated by $\tilde{\omega}/\mu$,
- the structure of the diffuse field is proportional to the phase function.

This expression also underlies the difficult of remote sensing scatterers using a nadir instrument. For example a monochromatic measurement from space of atmospheric particles whose phase function is well known is unable to differentiate between particles with a low $\tilde{\omega}$ and high optical depth and more reflective particles with a smaller optical depth.

6.5.1.1 Example: Using the Single Scattering Approximation to Calculate the Bond Albedo for a Homogeneous Spherical Atmosphere

A planet's Bond albedo, A, is the ratio of the total energy reflected by the illuminated hemisphere to the total solar energy incident upon the Earth. Mathematically this is

$$A = \frac{\int_{\lambda_1}^{\lambda_2}\int_0^{2\pi}\int_0^{\pi/2}\cos\theta_0 E_\lambda^{\text{Sun}} R(\lambda,\theta_0 : 2\pi) r_0^2 \sin\theta_0\, d\theta_0\, d\phi_0\, d\lambda}{\pi r_0^2 \int_{\lambda_1}^{\lambda_2} E_\lambda^{\text{Sun}}\, d\lambda} \quad (6.78)$$

where r_0 is the radius to the top of the atmosphere, λ_1 and λ_2 represent the limits of the solar spectral irradiance spectrum, θ_0 and ϕ_0 are the spherical coordinates for a global coordinate system with its origin at the Earth's centre and the z axis orientated

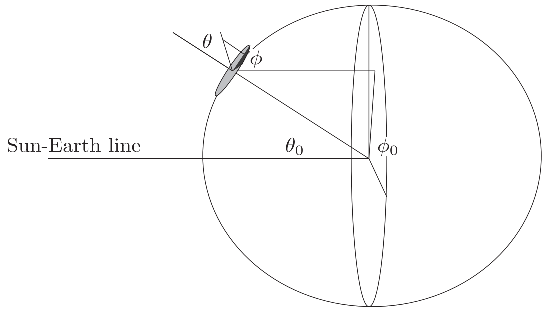

FIGURE 6.6
Global coordinate system.

to the sub-solar point (see Figure 6.6). Lastly $R(\lambda, \theta_0 : 2\pi)$ denotes the directional-hemispherical reflectance.

The optical properties of the layer are the single scatter albedo, $\tilde{\omega}$, the extinction coefficient, β^{ext}, and the phase function, $P(\Theta)$. Note that the input (θ_i, ϕ_i) and output (θ_o, ϕ_o) directions in spherical coordinates can be related to the scattering angle by

$$\cos\Theta = \cos\theta_i \cos\theta_o + \sin\theta_i \sin\theta_o \cos(\phi_o - \phi_i). \tag{6.79}$$

The choice of where $\phi = 0$ is arbitrary, so by setting it to ϕ_i the phase function can be shown as $P(\theta_i, \theta_0, \phi_o)$. The optical properties calculated by Mie theory are functions of wavelength, particle size and refractive index: however this dependence is not explicitly shown.

To calculate the change in Bond albedo it is necessary to introduce a local coordinate system (r, θ, ϕ) whose origin is at r_0, θ_0, ϕ_0 in global coordinates. The z axis of the local system points in the direction θ_0, ϕ_0. The choice of where $\phi = 0$ is arbitrary and is adopted here as the azimuthal direction of the solar ray.The directional-hemispherical reflectance from the atmosphere $R^{\text{l}}(\lambda, \theta_0 : 2\pi)$ is

$$R^{\text{l}}(\lambda, \theta_0 : 2\pi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\pi/2} \frac{\tilde{\omega}\tau P(\theta_0, \theta, \phi)}{4\pi\cos\theta\cos\theta_0}\cos\theta\sin\theta\, d\theta\, d\phi = \frac{\tilde{\omega}\tau}{\pi\cos\theta_0}\beta(\theta_0), \tag{6.80}$$

where τ is the optical depth of the layer and $\beta(\theta_0)$ is the upscatter fraction. The optical depth is related to the physical depth of the layer by

$$\tau(\lambda) = l\beta^{\text{ext}}(\lambda). \tag{6.81}$$

The upscatter fraction estimates the probability of a photon incident from direction θ_0 being lost to space [*Boucher*, 1998]. It is defined as

$$\beta(\theta_0) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi/2} P(\theta_0, \theta, \phi)\sin\theta\, d\theta\, d\phi, \tag{6.82}$$

where θ and ϕ are the local spherical coordinates. By making use of rotational symmetry and using the substitutions $\mu_0 = \cos\theta_0$ and $\mu = \cos\theta$, the Bond albedo is given by

$$A = \frac{2\int_{\lambda_1}^{\lambda_2} E_\lambda^{\text{Sun}} \tilde{\omega}\tau \int_0^1 \beta(\mu_0)\, d\mu_0\, d\lambda}{\int_{\lambda_1}^{\lambda_2} E_\lambda^{\text{Sun}}\, d\lambda} = \frac{2\int_{\lambda_1}^{\lambda_2} E_\lambda^{\text{Sun}} \tilde{\omega}\tau\overline{\beta}\, d\lambda}{\pi\int_{\lambda_1}^{\lambda_2} E_\lambda^{\text{Sun}}\, d\lambda}, \tag{6.83}$$

where

$$\overline{\beta} = \int_0^1 \beta(\mu_0)\, d\mu_0, \tag{6.84}$$

which is called the isotropic upscatter fraction.

6.5.2 Diffusion Approximation of Irradiance

The radiance within a non-scattering plane-parallel atmosphere where there are no extraterrestrial sources was considered in section 6.3.4. If there is no underlying surface contribution then the spectral irradiance at the top of an atmosphere of optical depth τ_0 is

$$E_{\tilde{\nu}}^{\uparrow}(0) = \int_0^{2\pi} \frac{1}{\mu} \int_0^{\tau_0} B_{\tilde{\nu}}(\tilde{\nu}, T(\tau)) e^{-\tau/\mu}\, d\tau\, d\Omega \tag{6.85}$$

As the blackbody emission is isotropic there is no azimuth dependence and this equation is rewritten as

$$E_{\tilde{\nu}}^{\uparrow}(0) = \int_0^{\tau_0} \pi B_{\tilde{\nu}}(\tilde{\nu}, T(\tau)) 2 \int_0^{\pi/2} e^{-\tau/\cos\theta} \cos\theta \sin\theta\, d\theta\, d\tau \tag{6.86}$$

In the diffuse approximation the integral over the set of slant paths is approximated by scaling the optical depth to give an expression of the form

$$E_{\tilde{\nu}}^{\uparrow}(0) = \int_0^{\tau_0^*} \pi B_{\tilde{\nu}}(\tilde{\nu}, T(\tau^*)) e^{-\tau^*}\, d\tau^* \tag{6.87}$$

where τ^* is the scaled optical depth such that $\tau^* = b\tau$ where b is a dimensionless constant. Comparing Equations 6.86 and 6.87 gives

$$\begin{aligned} e^{-\tau^*} &= 2b \int_0^{\pi/2} e^{-\tau/\cos\theta} \sin\theta\, d\theta \\ \Rightarrow \frac{b}{2} e^{-b\tau} &= \int_0^{\pi/2} e^{-\tau/\cos\theta} \sin\theta\, d\theta. \end{aligned}$$

If $\tau = 0$ then $b = 2$. For optical depths encountered in practice $b = 1.66$ gives a very good approximation. PLOT EXAMPLE

6.5.3 Two Stream Method

The vertical flux is a function of the zeroth order Fourier term

$$\mu \frac{dL_\nu(\tau;\mu)}{d\tau} = L_\nu(\tau;\mu) - \frac{\tilde{\omega}}{2}\int_{-1}^{1} p(\mu',\mu) L_\nu(\tau;\mu')\, d\mu'$$

where the azimuthally averaged phase function is

$$p(\mu',\mu) = \frac{1}{2\pi}\int_0^{2\pi} p(\mu',\mu,\Delta\phi)\, d\Delta\phi$$

In the two stream approximation we assume the radiation is constant in each hemisphere

$$L(\tau,\mu) = \begin{cases} L^{\uparrow}(\tau) & \mu > 0 \\ L^{\downarrow}(\tau) & \mu < 0 \end{cases}$$

For the upstream

$$\begin{aligned}
\mu \frac{dL_\nu^{\uparrow}(\tau)}{d\tau} &= L_\nu^{\uparrow}(\tau) - \frac{\tilde{\omega}}{2}\int_0^1 p(\mu',\mu) L_\nu^{\uparrow}(\tau)\, d\mu' - \frac{\tilde{\omega}}{2}\int_{-1}^{0} p(\mu',\mu) L_\nu^{\downarrow}(\tau)\, d\mu' \\
&= L_\nu^{\uparrow}(\tau) - \frac{\tilde{\omega}}{2} L_\nu^{\uparrow}(\tau)\int_0^1 p(\mu',\mu)\, d\mu' - \frac{\tilde{\omega}}{2} L_\nu^{\downarrow}(\tau)\int_{-1}^{0} p(\mu',\mu)\, d\mu' \\
&= L_\nu^{\uparrow}(\tau) - \tilde{\omega} L_\nu^{\uparrow}(\tau)(1 - b(\mu)) - \tilde{\omega} L_\nu^{\downarrow}(\tau) b(\mu)
\end{aligned}$$

where the azimuthally averaged phase function is

$$b(\mu) = \begin{cases} \frac{1}{2}\int_{-1}^{0} p(\mu',\mu)\, d\mu' = 1 - \frac{1}{2}\int_0^1 p(\mu',\mu)\, d\mu' & \mu > 0 \\ \frac{1}{2}\int_0^1 p(\mu',\mu)\, d\mu' = 1 - \frac{1}{2}\int_{-1}^{0} p(\mu',\mu)\, d\mu' & \mu < 0 \end{cases}$$

i.e. $b(\mu)$ represents the fraction of radiation that is scattered into the opposite hemisphere. The dependence of b on μ is unhelpful so we just integrate the equation over μ and work in irradiance

$$\int_0^1 \left[\mu \frac{dL_\nu^{\uparrow}(\tau)}{d\tau}\right] d\mu = \int_0^1 \left[L_\nu^{\uparrow}(\tau) - \tilde{\omega} L_\nu^{\uparrow}(\tau)(1 - b(\mu)) - \tilde{\omega} L_\nu^{\downarrow}(\tau) b(\mu)\right] d\mu$$

$$\begin{aligned}
\frac{1}{2}\frac{dE_\nu^{\uparrow}(\tau)}{d\tau} &= E_\nu^{\uparrow}(\tau) - \tilde{\omega} E_\nu^{\uparrow}(\tau)(1 - \overline{b}) - \tilde{\omega} E_\nu^{\downarrow}(\tau)\overline{b} \\
&= (1 - \tilde{\omega}) E_\nu^{\uparrow}(\tau) + \tilde{\omega}\overline{b}(E_\nu^{\uparrow}(\tau) - E_\nu^{\downarrow}(\tau))
\end{aligned}$$

where

$$\overline{b} = \int_0^1 b(\mu)\, d\mu \qquad \text{noting} \begin{array}{c} \overline{b} = 1 \Rightarrow g = -1 \\ \overline{b} = 1/2 \Rightarrow g = 0 \\ \overline{b} = 0 \Rightarrow g = 1 \end{array} \text{ so } \overline{b} = \frac{1 - g}{2}$$

Similarly

$$-\frac{1}{2}\frac{dE_\nu^\downarrow(\tau)}{d\tau} = (1-\tilde{\omega})E_\nu^\downarrow(\tau) - \tilde{\omega}\overline{b}(E_\nu^\uparrow(\tau) - E_\nu^\downarrow(\tau))$$

Finally adding and subtracting gives

$$\frac{1}{2}\frac{d}{d\tau}(E_\nu^\uparrow(\tau) - E_\nu^\downarrow(\tau)) = (1-\tilde{\omega})(E_\nu^\uparrow(\tau) + E_\nu^\downarrow(\tau))$$
$$\frac{1}{2}\frac{d}{d\tau}(E_\nu^\uparrow(\tau) + E_\nu^\downarrow(\tau)) = \tilde{\omega}(1-g)(E_\nu^\uparrow(\tau) - E_\nu^\downarrow(\tau))$$

6.5.3.1 Example: Using the Two Stream Method to Calculate the Reflectance and Transmittance of a non-absorbing Cloud

If $\tilde{\omega} = 1$ then

$$\frac{1}{2}\frac{d}{d\tau}(E_\nu^\uparrow(\tau) - E_\nu^\downarrow(\tau)) = 0$$
$$\Rightarrow E_\nu^\uparrow(\tau) - E_\nu^\downarrow(\tau) = E^{\text{net}}$$

where E^{net} is the constant of integration and equal to the net flux which does not change with depth in the cloud. Similarly

$$\frac{1}{2}\frac{d}{d\tau}(E_\nu^\uparrow(\tau) + E_\nu^\downarrow(\tau)) = (1-g)E^{\text{net}}$$
$$\Rightarrow E_\nu^\uparrow(\tau) + E_\nu^\downarrow(\tau) = 2(1-g)E^{\text{net}}\tau + K$$

where K is a constant of integration. Together the two equations give:

$$E_\nu^\uparrow(\tau) = \frac{E^{\text{net}}}{2}(1 + 2\tau(1-g)) + \frac{K}{2}$$
$$E_\nu^\downarrow(\tau) = -\frac{E^{\text{net}}}{2}[1 - 2\tau(1-g)] + \frac{K}{2}$$

Apply boundary conditions

$$E_\nu^\downarrow(0) = \mu^{\text{Sun}}E_0 \quad E_\nu^\uparrow(\tau_c) = 0$$

gives

$$\frac{K}{2} = \mu^{\text{Sun}}E_0 + \frac{E^{\text{net}}}{2} \quad E^{\text{net}} = \frac{-\mu^{\text{Sun}}E_0}{1+(1-g)\tau_c} \tag{6.88}$$

General solutions

$$E_\nu^\uparrow(\tau) = \frac{\mu^{\text{Sun}}E_0(1-g)(\tau_c-\tau)}{1+(1-g)\tau_c} \quad E_\nu^\downarrow(\tau) = \frac{\mu^{\text{Sun}}E_0\,[1+(1-g)(\tau_c-\tau)]}{1+(1-g)\tau_c}$$

From which it is easy to show

$$r = \frac{E_\nu^\uparrow(0)}{\mu^{\text{Sun}}E_0} = \frac{(1-g)\tau_c}{1+(1-g)\tau_c} \quad t = \frac{E_\nu^\downarrow(\tau_c)}{\mu^{\text{Sun}}E_0} = \frac{1}{1+(1-g)\tau_c}$$

Problem 6.1 Develop an expression for the radiation at the top of an isothermal atmosphere where scattering contributes to the reduction in the transmitted beam but makes a negligible contribution to the transmitted radiation.

Problem 6.2 The Fourier expansion of the diffuse source function is

$$J_\lambda^{\text{diffuse}}(\tau, \omega) = \sum_{m=0}^{\infty} J_\lambda^m(\tau, \mu) \cos m(\phi - \phi_0)$$

where each term is given by

$$J_\lambda^m(\tau, \mu) = (2 - \delta_{0,m}) \frac{1}{2\pi} \int_0^{2\pi} J_\lambda^{\text{diffuse}}(\tau, \omega) \cos m(\phi - \phi_0)\, d\phi$$

Starting from this expression derive Equation 6.50.

Problem 6.3 By substituting the Fourier expression for $L(\omega)$ into the expression for irradiance show that the upward irradiance $E^{\uparrow}$ is only a function of the 0^{th}order Fourier radiance $L_\lambda^0(\tau, \mu)$ (i.e. derive Equation 6.38).

Additional Reading

Lenoble, J., *Radiative Transfer in Scattering and Absorbing Atmospheres*, A. Deepak, Hampton, Virginia, USA, 1985.