A

Tables of Constants

TABLE A.1
Physical Constants

Avogadro's number	N_a	$6.022\,141\,79 \times 10^{23}$ molecule mol^{-1}
Boltzmann's constant	k_B	$1.380\,650\,4$ J K^{-1}
Electron charge	e	$1.602\,176\,487 \times 10^{-19}$ C
Gravitational constant	G	$6.674\,29 \times 10^{-11}$ m^3 kg^{-1} s^{-2}
Magnetic permeability of free space	μ_0	$4\pi \times 10^{-7}$ kg m s^{-2} A^{-2}
Mass of an electron	m_e	$9.109\,382\,15 \times 10^{-31}$ kg
Permittivity of free space	ϵ_0	$8.854\,187\,817 \times 10^{-7}$ kg^{-1} m^{-3} s^4 A^2
Planck's constant	h	$6.626\,068\,96 \times 10^{-34}$ J s
Stefan-Boltzmann constant	σ	$5.670\,400 \times 10^{8}$ J m^{-2} s^{-1} K^{-4}
Speed of light in a vacuum	c	$2.997\,924\,58 \times 10^{8}$ m s^{-1}
Universal gas constant	R	$8.314\,472$ J mol^{-1} K^{-1}

TABLE A.2
Astronomical Values

Acceleration of gravity (at sea level and 45 °latitude)	g	$9.80616\ \mathrm{m\,s^{-2}}$
Standard gravity (nominal global average)	g	$9.80665\ \mathrm{m\,s^{-2}}$
Angular velocity of rotation of the Earth	ω	$7.27221 \times 10^{5}\ \mathrm{rad\,s^{-1}}$
Average distance, Sun to Earth	D_S	1.496×10^{8} km
Average distance, Earth to Moon	D_M	3.84×10^{5} km
Radius of the Earth (volumetric mean)	R_E	6371 km
Average radius of the Moon	R_M	1740 km
Average radius of the Sun (visible disk)	R_S	6.96×10^{5} km
Average solar flux at TOA	E^0	$1366\ \mathrm{W\,m^{-2}}$
Mass of the earth	M_E	5.988×10^{24} kg

TABLE A.3
Relevant Meteorological Constants

Density of air at standard pressure and temperature	ρ	$1.273 \times 10^{-3}\ \mathrm{g\,cm^{-3}}$; $1.273\ \mathrm{kg\,m^{-3}}$
Density of ice (0°C)	ρ_i	$0.917 \times 10^{3}\mathrm{kg\,m^{-3}}$
Density of liquid water (4°C)	ρ_l	$1 \times 10^{3}\mathrm{kg\,m^{-3}}$
Dry air gas constant	R_{air}	$287\ \mathrm{J\,kg^{-1}K^{-1}}$
Latent heat of fusion, ice	L_{ice}	$3.34 \times 10^{5}\ \mathrm{J\,kg^{-1}}$
Latent heat of vaporization, water at 0 °C	L	$2.50 \times 10^{6}\ \mathrm{J\,kg^{-1}}$
Molecular weight of dry air	M	$28.97\ \mathrm{g\,mol^{-1}}$
Saturation vapour pressure (0 °C)	e_0	6.1078 mb
Specific heat of air at constant pressure	C_p	$10.04 \times 10^{2}\ \mathrm{J\,kg^{-1}K^{-1}}$
Specific heat of air at constant volume	C_v	$7.17 \times 10^{2}\ \mathrm{J\,kg^{-1}K^{-1}}$
Standard pressure	p_0	1013.25 mb 1.013×10^{5} Pa
Standard temperature	T_0	273.16 K

B

Mathematical Definitions and Identities

B.1 Mathematical Operators in Cartesian Coordinates

del

$$\nabla f = \frac{\partial f_x}{\partial x}\mathbf{i}_x + \frac{\partial f_y}{\partial y}\mathbf{i}_y + \frac{\partial f_z}{\partial z}\mathbf{i}_z \tag{B.1}$$

divergence

$$div f = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \tag{B.2}$$

$$= \nabla \cdot f \tag{B.3}$$

curl Laplacian

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{B.4}$$

B.2 Matrix Algebra

For martices **A** and **B**

$$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} \tag{B.5}$$

For a symmetric matrix **S**

$$\frac{\partial}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x}_{\mathrm{a}})^{\mathrm{T}}\mathbf{S}^{-1}(\mathbf{x} - \mathbf{x}_{\mathrm{a}}) = 2\mathbf{S}(\mathbf{x} - \mathbf{x}_{\mathrm{a}}) \tag{B.6}$$

$$\frac{\partial}{\partial \mathbf{x}}(\mathbf{y} - \mathbf{Kx})^{\mathrm{T}}\,\mathbf{S}^{-1}\,(\mathbf{y} - \mathbf{Kx}) = -2\mathbf{K}^{\mathrm{T}}\mathbf{S}(\mathbf{y} - \mathbf{Kx}) \tag{B.7}$$

B.3 Legendre Polynomials

The Legendre polynomials $P_l(x), l = 0, 1, 2, \ldots,$ are solutions of Legendre's differential equation

$$(1 - x^2)\frac{\partial^2 y}{\partial x^2} - 2x\frac{\partial y}{\partial x} + l(l+1)y = 0 \tag{B.8}$$

when x is a real number in the range $[-1, 1]$. For $l \geq 0$ the polynomial can be written

$$P_l(x) = \sum_{r=0}^{m} \frac{(2l-2r)!}{2^l r!(l-r)!(l-2r)!} x^{l-2r} \tag{B.9}$$

where the integer m is $l/2$ or $(l-1)/2$. The explicit forms of the first six Legendre polynomials are

$$\begin{aligned} &P_0(x) = 1, && P_1(x) = x, \\ &P_2(x) = \tfrac{1}{2}(3x^2 - 1), && P_3(x) = \tfrac{1}{2}(5x^3 - 3x), \\ &P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3), && P_5(x) = \tfrac{1}{8}(65x^5 - 70x^3 + 15x). \end{aligned} \tag{B.10}$$

The polynomials can be evaluated at the specific arguments $-1, 0$ and 1 to be

$$P_l(-1) = (-1)^l \tag{B.11}$$

$$P_l(0) = \frac{\cos(l\pi/2)\Gamma[(l+1)/2]}{\sqrt{\pi}\Gamma(l/2+1)} \tag{B.12}$$

$$P_l(1) = 1 \tag{B.13}$$

Recurrence relations can be used to evaluate higher order Legendre polynomials. Formulae include

$$P'_{l+1}(x) = (2l+1)P_l(x) + P'_{l-1}(x) \tag{B.14}$$

$$(l+1)P_{l+1}(x) = (2l+1)xP_l(x) - lP_{l-1}(x) \tag{B.15}$$

$$P'_{l+1}(x) = xP'_l(x) + (l+1)P_l(x) \tag{B.16}$$

$$(x^2 - 1)P'_l(x) = lxP_l(x) - lP_{l-1}(x) \tag{B.17}$$

$$P'_l(x) = \frac{l(l+1)}{2l+1}\frac{1}{1-x^2}[P_{l+1}(x) - P_{l-1}(x) \tag{B.18}$$

where the prime denotes differentiation with respect to x. Finally the Legendre polynomials are interrelated through

$$P_l{-x} = (-1)^l P_l(x) \tag{B.19}$$

$$P_{-l-1}(x) = P_l(x) \tag{B.20}$$

B.4 Associated Legendre Polynomials

The equation for the associated Legendre polynomials is [*Abramowitz and Stegun*, 1964]

$$(1-x^2)\frac{\partial^2 y}{\partial x^2} - 2x\frac{\partial y}{\partial x} + \left[l(l+1) - \frac{m^2}{1-x^2}\right] y = 0 \tag{B.21}$$

has a solution ([redacted] t [redacted] he solution needs a reference, also need to discuss arguments greater than 1)

$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^m P_l(x)}{dx^m} \tag{B.22}$$

where x is restricted to $[-1, 1]$. Some authors include a factor $(-1)^m$ in the definition of $P_l^m(x)$. The associated Legendre polynomials have the following explicit forms

$$\begin{array}{lll} P_0^0(x) = 1 & & \\ P_1^0(x) = x & P_1^1(x) = (1-x^2)^{1/2} & \\ P_2^0(x) = \frac{1}{2}\left(3x^2-1\right) & P_2^1(x) = 3x(1-x^2)^{1/2} & P_2^2(x) = 3(1-x^2) \\ P_3^0(x) = \frac{1}{5}\left(5x^3-3x\right) & P_3^1(x) = \frac{3}{2}\left(5x^3-1\right)\left(1-x^2\right)^{1/2} & P_3^2(x) = 15(1-x^2) \end{array} \tag{B.23}$$

also

$$P_l^0(x) = P_l(x) \quad \text{and} \quad P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x). \tag{B.24}$$

The associated Legendre polynomials obey the following recurrence relations

$$(2l+1)xP_l^m(x) = (l+m)P_{l-1}^m(x) + (l-m+1)P_{l+1}^m(x), \tag{B.25}$$

$$(1-x^2)^{1/2}\frac{dP_l^m(x)}{dx} = \frac{1}{2}(l+m)(l-m+1)P_l^{m-1}(x) + \frac{1}{2}P_l^{m+1}(x), \tag{B.26}$$

$$(x^2-1)P_l^m(x) = lxP_l^m(x) - (l+m)P_{l-1}^m(x). \tag{B.27}$$

B.5 Mie Angular Functions

The Mie angular functions π_n and τ_n are defined in terms of he associated Legendre polynomials as

$$\pi_n(\cos\Theta) = \frac{1}{\sin\Theta}P_n^1(\cos\Theta) \tag{B.28}$$

$$\tau_n(\cos\Theta) = \frac{d}{d\Theta}P_n^1(\cos\Theta). \tag{B.29}$$

Evaluating the first terms gives

$$\begin{array}{lll} \pi_0(\cos\Theta) = 0, & \pi_1(\cos\Theta) = 1, & \pi_2(\cos\Theta) = 3\cos\Theta \\ \tau_0(\cos\Theta) = 0, & \tau_1(\cos\Theta) = \cos\Theta, & \tau_2(\cos\Theta) = 3\cos 2\Theta \end{array} \tag{B.30}$$

$$\pi_n(\cos\Theta) = \frac{1}{\sin\Theta}\frac{(2n-1)\cos\Theta P_{n-1}(\cos\Theta) - (n-1)P_{n-2}(\cos\Theta)}{n} \quad \text{(B.31)}$$

$$\frac{1}{\sin\Theta}\frac{(2n-1)\cos\Theta P_{n-1}(\cos\Theta)}{n} - \frac{(n-1)P_{n-2}(\cos\Theta)}{n} \quad \text{(B.32)}$$

B.6 Bessel Functions

The solutions to Bessel's Equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \quad \text{(B.33)}$$

are called Bessel functions. The order of the Bessel function solution is determined by the constant n. When n is not an integer the two solutions of Equation B.33 are the Bessel function of the first kind of order n, $J_n(x)$. Expressed as a series [see *Boas*, 2006, for the derivation] they are

$$J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(m+1)\Gamma(m+1+n)}\left(\frac{x}{2}\right)^{2m+n} \quad \text{(B.34)}$$

and

$$J_{-n}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(m+1)\Gamma(m+1-n)}\left(\frac{x}{2}\right)^{2n-n} \quad \text{(B.35)}$$

giving the general solution to Bessel's equation of

$$y(x) = c_1J_n(x) + c_2J_{-n}(x) \qquad n \text{ not an integer} \quad \text{(B.36)}$$

where c_1 and c_2 are constants.

To provide solutions at integer orders a combination of $J_n(x)$ and $J_{-n}(x)$ are used as the second solution of Bessel's equation. These are $Y_n(x)$ called Bessel functions of the second kind of order n and defined

$$Y_n(x) = \frac{\cos\pi n J_n(x) - J_{-n}(x)}{\sin\pi n}. \quad \text{(B.37)}$$

In some texts Bessel functions of the second kind are called the Weber functions or the Neumann functions (and so written $N_n(x)$). The general solution of Bessel's equation applicable for all n is then

$$y(x) = c_3J_n(x) + c_4Y_n(x), \qquad \text{all } n \quad \text{(B.38)}$$

where c_3 and c_4 are constants.

Finally a complex linear combination of Bessel functions of the first and second kinds is used to form Bessel functions of the third kind (also known as Hankel functions) $H_n^{(1)}$ and $H_n^{(2)}$. These are defined

$$H_n^{(1)}(x) = J_n(x) + iY_n(x), \tag{B.39}$$

$$H_n^{(2)}(x) = J_n(x) - iY_n(x). \tag{B.40}$$

Boas [2006] develops the following relations between the Bessel functions and their derivatives (which are also valid for the Neumann function),

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x), \tag{B.41}$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x), \tag{B.42}$$

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x), \tag{B.43}$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x), \tag{B.44}$$

$$J_n'(x) = -\frac{n}{x} J_n(x) + J_{n-1}(x) = \frac{n}{x} J_n(x) - J_{n+1}(x). \tag{B.45}$$

B.7 Spherical Bessel Functions

The spherical Bessel function are defined in terms the Bessel functions of half-odd integer order. They are [*Boas*, 2006]

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x), \tag{B.46}$$

$$y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+1/2}(x), \tag{B.47}$$

$$h_n^1(x) = j_n(x) + iy_n(x) = \sqrt{\frac{\pi}{2x}} H_{n+1/2}^{(1)}(x), \tag{B.48}$$

$$h_n^2(x) = j_n(x) - iy_n(x) = \sqrt{\frac{\pi}{2x}} H_{n+1/2}^{(2)}(x). \tag{B.49}$$

The power series expansions for the first and second order spherical Bessel functions are

$$j_n(x) = 2^n x^n \sum_{m=0}^{\infty} \frac{(-1)^m (n+m)!}{m!(2n+2m+1)!} x^{2m}, \tag{B.50}$$

$$y_n(x) = -\frac{1}{2^n x^{n+1}} \sum_{m=0}^{\infty} \frac{\Gamma(2n-2m+1}{m!\Gamma(n-m+1)} z^{2m}. \tag{B.51}$$

Evaluating these expressions for the first few terms gives

$$\begin{array}{lll} j_0(x) = 1 - \frac{x^2}{6} + \frac{x^4}{120}, & j_1(x) = \frac{x}{3} - \frac{x^3}{30} + \frac{x^5}{840}, & j_2(x) = \frac{x^2}{15} - \frac{x^4}{210} + \frac{x^6}{3780}, \\ y_0(x) = -\frac{1}{x} + \frac{x}{2} - \frac{x^3}{24}, & y_1(x) = -\frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{4}, & y_2(x) = -\frac{3}{x^3} - \frac{1}{2x} - \frac{x}{8}. \end{array} \tag{B.52}$$

B.8 Ricatti-Bessel Functions

The Ricatti-Bessel functions $\psi_n(x)$, $\chi_n(x)$ and $\zeta_n(x)$ are the product of the relevant spherical Bessel function with its argument, i.e.

$$\psi_n(x) = x j_n(x) = \sqrt{\frac{\pi x}{2}} J_{n+1/2}(x), \tag{B.53}$$

$$\chi_n(x) = x y_n(x) = \sqrt{\frac{\pi x}{2}} Y_{n+1/2}(x), \tag{B.54}$$

$$\zeta_n(x) = \psi_n(x) + i\chi_n(x) = \sqrt{\frac{\pi x}{2}} H^{(1)}_{l+1/2}(x). \tag{B.55}$$

The infinite series expressions are

$$\psi_n(x) = 2^n x^{n+1} \sum_{m=0}^{\infty} \frac{(-1)^m (n+m)!}{m!(2n+2m+1)!} x^{2m}, \tag{B.56}$$

$$\chi_n(x) = -\frac{1}{2^n x^n} \sum_{m=0}^{\infty} \frac{\Gamma(2n-2m+1)}{m!\Gamma(n-m+1)} x^{2m}. \tag{B.57}$$

Alternatively they can be written as a terminating series [*Gumprecht and Sliepcevich*, 1951]

$$\begin{aligned} \psi_n(x) = {} & \sin\left(x - \frac{n\pi}{2}\right) \sum_{m=0}^{\leq n/2} \frac{(-1)^m (n+2m)!}{(2m)!(n-2m)!(2x)^{2m}}, \\ & + \cos\left(x - \frac{n\pi}{2}\right) \sum_{m=0}^{\leq (n-1)/2} \frac{(-1)^m (n+2m+1)!}{(2m+1)!(n-2m-1)!(2x)^{2m+1}}, \end{aligned} \tag{B.58}$$

$$\begin{aligned} \chi_n(x) = {} & (-1)^n \cos\left(x + \frac{n\pi}{2}\right) \sum_{m=0}^{\leq n/2} \frac{(-1)^m (n+2m)!}{(2m)!(n-2m)!(2x)^{2m}}, \\ & - \sin\left(x - \frac{n\pi}{2}\right) \sum_{m=0}^{\leq (n-1)/2} \frac{(-1)^m (n+2m+1)!}{(2m+1)!(n-2m-1)!(2x)^{2m+1}}. \end{aligned} \tag{B.59}$$

Using these expressions the Ricatti-Bessel functions for the first two orders are

$$\begin{array}{lll} \psi_0(x) = \sin x & \chi_0(x) = \cos x & \zeta_0(x) = \sin x + i \cos x \\ \psi_1(x) = \cos x & \chi_1(x) = -\sin x & \zeta_1(x) = \cos x - i \sin x \end{array} \tag{B.60}$$

Higher order values can be calculated using the recurrence relation

$$\zeta_n(x) = \frac{2n-1}{x}\zeta_{n-1}(x) - \zeta_{n-2}(x) \tag{B.61}$$

C

Series Expansion of the Solution of the Spherical Wave Equation

The three dimensional wave or Helmholtz equation

$$\nabla^2\Pi + k^2\Pi = 0 \tag{C.1}$$

can be expressed in spherical coordinates as [*Kerker*, 1969]

$$\frac{1}{r}\frac{\partial^2(r\Pi)}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Pi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Pi}{\partial\phi^2} + k^2\Pi = 0 \tag{C.2}$$

Using the method of separation of variables, a solution of the form

$$\Pi = R(r)\Theta(\theta)\Phi(\phi) \tag{C.3}$$

is adopted. Substituting this into Equation C.2 and multipling by $r^2/R\Theta\Phi$ gives

$$\frac{r}{R}\frac{\partial^2(rR)}{\partial r^2} + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} + r^2k^2 = 0 \tag{C.4}$$

which can be multiplied by $\sin^2\theta$ so that the third term is purely a function of ϕ. Separating the equation gives

$$\left[\frac{r}{R}\frac{\partial^2(rR)}{\partial r^2} + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + r^2k^2\right]\sin^2\theta = m^2, \tag{C.5}$$

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = -m^2. \tag{C.6}$$

The requirement of a single value solution implies that the solution for Φ must have the same value for ϕ as for $\phi + 2m\pi$ where m is an integer. Applying this restriction means that the solution for Equation C.6 is

$$\Phi = a_m\cos(m\phi) + b_m\sin(m\phi)\,. \tag{C.7}$$

where a_m and b_m are constants.

Equation C.5 can be rewritten as

$$\frac{r}{R}\frac{\partial^2(rR)}{\partial r^2} + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + r^2k^2 - \frac{m^2}{\sin^2\theta} = 0 \tag{C.8}$$

so that by introducing a further constant p, it can be separated into

$$\frac{r}{R}\frac{\partial^2(rR)}{\partial r^2} + r^2k^2 = p \tag{C.9}$$

$$\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta} = -p \tag{C.10}$$

Substituting $x = \cos\theta$ reveals that Equation C.10 is the equation for the associated Legendre functions [*Abramowitz and Stegun*, 1964]

$$(1-x^2)\frac{\partial^2\Theta}{\partial x^2} - 2x\frac{\partial\Theta}{\partial x} + \left[l(l+1) - \frac{m^2}{1-x^2}\right]\Theta = 0 \tag{C.11}$$

given that $p = l(l+1)$ and l must be a integer in order that the solutions are finite at $x = \cos\theta = \pm 1$. The solution of C.10 is then

$$\Theta = P_l^m(\cos\theta). \tag{C.12}$$

where P_l^m are the associated Legendre functions. They can be evaluated by means of recursion relationships defined in Appendix B.

The radial expression (Equation C.9) is now expressed as

$$\frac{\partial^2(rR)}{\partial r^2} + \left[k^2 - \frac{l(l+1)}{r^2}\right]rR = 0 \tag{C.13}$$

and solved by making the substitutions

$$kr = \rho \quad \text{and} \quad R(r) = \frac{1}{\sqrt{\rho}}Z(\rho) \tag{C.14}$$

to obtain the Bessel equation of half integer order

$$\rho^2\frac{\partial^2 Z}{\partial\rho^2} + \rho\frac{\partial Z}{\partial\rho} + \left[\rho^2 - \left(l + \frac{1}{2}\right)^2\right]Z = 0 \tag{C.15}$$

The two solutions of this expression are the half integral order Bessel function $J_{l+1/2}(\rho)$ and the half integral order Neumann function $N_{l+1/2}(\rho)$. The general solution of Bessel's equation [*Boas*, 2006] may be written as

$$Z(\rho) = AJ_{l+1/2}(\rho) + BN_{l+1/2}(\rho) \tag{C.16}$$

where A and B are arbitrary constants. The solution of Equation C.9 to be written as

$$R(r) = \frac{1}{\sqrt{kr}}\left[AJ_{l+1/2}(kr) + BN_{l+1/2}(kr)\right] \tag{C.17}$$

Choosing different constants c_l and d_l defined by

$$c_l = \frac{A}{k}\sqrt{\frac{2}{\pi}} \quad \text{and} \quad d_l = -\frac{B}{k}\sqrt{\frac{2}{\pi}} \tag{C.18}$$

allows Equation C.17 to be expressed as

$$rR(r) = c_l \sqrt{\frac{\pi kr}{2}} J_{l+1/2}(kr) - d_l \sqrt{\frac{\pi kr}{2}} N_{l+1/2}(kr) \tag{C.19}$$

$$= c_l \psi_l(kr) + d_l \chi_l(kr) \tag{C.20}$$

where $\psi_l(\rho)$ and $\chi_l(\rho)$ are the Ricatti-Bessel functions which are defined in terms of the Bessel and Neumann functions [*Kerker*, 1969] by

$$\psi_l(\rho) = \sqrt{\frac{\pi\rho}{2}} J_{l+1/2}(\rho), \tag{C.21}$$

$$\chi_l(\rho) = -\sqrt{\frac{\pi\rho}{2}} N_{l+1/2}(\rho). \tag{C.22}$$

Hence

$$rR = c_l \psi_l(kr) + d_l \chi_l(kr) \tag{C.23}$$

The Ricatti-Bessel functions can be evaluated by means of recursion relationships defined in Appendix B. The functions $\chi_l(kr)$ become infinite at the origin so can not be used to represent a wave where r can equal 0.

The general solution of the wave equation in spherical coordinates is obtained from a linear superposition of all the particular solutions

$$\begin{aligned} r\Pi &= r \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \pi_n^m \\ &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[c_l \psi_l(kr) + d_l \chi_l(kr)\right] P_l^m(\cos\theta) \left[a_m \cos(m\phi) + b_m \sin(m\phi)\right] \end{aligned} \tag{C.24}$$

References

References

Abramowitz, M., and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1964.

Allen, A., Expansion of the mie phase function in legendre polynomials, *Applied Optics*, *13*, 2752–2753, 1974.

Atkins, P., and R. S. Friedman, *Molecular Quantum Mechanics*, Oxford University Press, New York, 2005.

Banwell, C. N., and E. M. McCash, *Fundamentals of Molecular Spectroscopy*, McGraw- Hill, Maidenhead, 1994.

BIPM, *The International System of Units (SI)*, 8th ed., Bureau International des Poids et Mesures, 2006.

BIPM, *Evaluation of measurement data — Guide to the Expression of Uncertainty in Measurement*, 1st ed., Bureau International des Poids et Mesures, Sèvres, France, 2008.

Boas, M., *Mathematical Methods in the Physical Sciences*, Wiley, New York, 2006.

Bohr, N., On the constitution of atoms and molecules, Part I, *Philosophical Magazine*, *26*, 1–24, 1913.

Bohren, C. F., and D. R. Huffman, *Absorption and Scattering of Light by Small Particles*, Wiley, New York, 1983.

Boucher, O., On aerosol direct shortwave forcing and the Henyey-Greenstein phase function, *J. Atmos. Sci.*, *55*, 128–134, 1998.

Brooker, G., *Modern Classical Optics*, Harper Perennial, New York, 2003.

Chahine, M. T., Interaction mechanisms within the atmosphere, in *Manual of Remote Sensing*, edited by R. N. Colwell, American Society of Photogrammetry, Falls Church, Virginia, 1983.

Chu, C., and S. W. Churchill, Representation of the angular distribution coefficients of radiation scattered by a spherical particle, *Journal of the Optical Society of America*, *45*, 958–962, 1955.

Clark, G. C., C. Chu, and S. W. Churchill, Angular disribution coefficients for radiation scattered by a spherical particle, *Journal of the Optical Society of America*, *47*, 81–84, 1957.

Dave, J. V., and J. Gazdag, A modified Fourier transform method for multiple scattering calculations in a plane-parallel Mie atmosphere, *Applied Optics*, *9*, 1457–1466, 1970.

Duffin, W., *Electricity and Magnetism*, McGraw-Hill, New York, 1990.

Elsasser, W. M., Mean absorption and equivalent absorption coefficient of a band spectrum, *Physical Review, 54*(2), 126–129, 1938.

Fenn, R. W., S. A. Clough, W. O. Gallery, R. E. Good, F. X. Kneizys, J. D. Mill, L. S. Rothman, E. P. Shettle, and F. E. Volz, ???, in *Handbook of Geophysics and the Space Environment*, edited by A. S. Jursa, Air Force Geophysics Laboratory, Air Force Systems Command, United States Air Force, 1985.

Fowler, B., Expansion of mie-theory phase functions in series of legendre polynomials, *Journal of the Optical Society of America, 73*, 19–22, 1983.

Fowles, G. R., *Introduction to Modern Optics*, Dover, 1989.

Fröhlich, C., and J. Lean, Solar radiative output and its variability: evidence and mechanisms, *The Astronomy and Astrophysics Review*, p. 273320, 2004.

Goody, R. M., A statistical model for water-vapour absorption, *Quarterly Journal of the Royal Meteorological Society*, *78*(336), 165–169, 1952.

Goody, R. M., *Atmospheric Radiation I: Theoretical basis*, Clarendon Press, Oxford, 1964.

Grandy, W., *Scattering of Waves from Large Spheres*, Cambridge University Press, Cambridge, 2000.

Gumprecht, R. O., and C. M. Sliepcevich, *Tables of Riccati Bessel functions for large arguments and orders*, Engineering Reserach Institute, University of Michigan (Ann Arbor), 1951.

Hansen, J., and J. B. Pollack, Near-infrared light scattering by terrestrial clouds, *Journal of the Atmospheric Sciences*, *27*, 265–281, 1970.

Henyey, L. C., and J. L. Greenstein, Diffuse radiation in the galaxy, *The Astophysical Journal*, *93*, 70–83, 1941.

Hunt, G., The generation of angular distribution coefficients for radiation scattered by a spherical particle, *Journal of Quantitative Spectroscopy and Radiative Transfer*, *10*, 857–864, 1970.

Iqbal, M., *An Introduction to Solar Radiation*, Academic Press, Canada, 1983.

ISO, *ISO 31-6:1992/Amd.1:1998, Quantities and units, Part 6: Light and related electromagnetic radiations*, International Organization for Standardization, 1992.

Jackson, J., *Classical Electrodynamics*, Wiley, New York, 1999.

Kattawar, G., S. J. Hitzfelder, and J. Binstock, An explicit form of the mie phase matrix for multiple scattering calculations in the I, Q, U and V representation, *Journal of the Atmospheric Sciences*, *30*, 289–295, 1973.

Kerker, M., *The Scattering of Light*, Academic Press, 1969.

King, M. D., Number of terms required in the Fourier expansion of the reflection function for optically thick atmospheres, *Journal of Quantitative Spectroscopy and Radiative Transfer*, *30*, 143–161, 1983.

Lenoble, J., *Radiative Transfer in Scattering and Absorbing Atmospheres*, A. Deepak, Hampton, Virginia, USA, 1985.

Liou, K.-N., *An Introduction To Atmospheric Radiation*, Academic Press, New York, 1980.

McCluney, R., *Introduction to Radiometry and Photometry*, Artech House, Boston, 1994.

McKellar, B. H. J., and M. A. Box, The scaling group of the radiative transfer equation, *Journal of the Atmospheric Sciences*, *34*, 1063–1068, 1981.

Mohr, P. J., B. N. Taylor, and D. B. Newell, CODATA recommended values of the fundamental physical constants: 2006, *Reviews of Modern Physics*, *80*(2), 633–730, 2008.

Paltridge, G. W., and C. M. R. Platt, *Radiative Processes in Meteorology and Climatology*, Elsevier, Amsterdam, 1976.

Pedrotti, F. L., L. S. Pedrotti, and L. M. Pedrotti, *Introduction to Optics*, third ed., Pearson Education, Upper Saddle River, 2007.

Petty, G., *A First Course in Atmospheric Radiation*, Oxford University Press, Oxford, 2004.

Press, W., *Numerical Recipes in Fortran*, Cambridge University Press, Cambridge, 1992.

Rodgers, C. D., Retrieval of atmospheric temperature and composition from remote measurements of thermal radiation, *Review of Geophysics and Space Physics*, *14*, 609, 1976.

Rodgers, C. D., *Inverse Methods for Atmospheric Sounding: Theory and Practice*, World Scientific Pub Co Inc, Singapore, 2000.

Schaepman-Strub, G., M. E. Schaepman, T. H. Painter, S. Dangel, and J. V. Martonchik, Reflectance quantities in optical remote sensing — definitions and case studies, *Remote Sensing of Environment*, *103*(1), 27–42, 2006.

Schröder, K. P., and R. C. Smith, Distant future of the sun and earth revisited, *Monthly Notices of the Royal Astronomical Society*, *386*, 155–163, 2008.

Shannon, C., and W. Weaver, *The Mathematical Theory of Communication*, University of Illinois Press, Urbana, Illinois, 1949.

Slater, P. N., *Remote Sensing Optics and Optical Systems*, Addison-Wesley Publishing Company, London, 1980.

Thorne, A., *Spectrophysics: Principles and Applications*, Springer, Berlin, 1988.

Thuillier, G., M. Hersé, P. C. Simon, D. Labs, H. Mandel, D. Gillotay, and T. Foujols, The solar spectral irradiance from 200 to 2400 nm as measured by the SOLSPEC spectrometer from the ATLAS 1-2-3 and EURECA missions, *Solar Physics*, *214*, 1–22, 2003.

Twomey, S., *Introduction to the Mathematics of Inversion in Remote Sensing and Indirect Measurements*, Dover Publications, New York, 1996.

van de Hulst, H. C., *Multiple Light Scattering*, Academic Press, 1980.

van Vleck, J. H., and V. F. Weisskopf, On the shape of collision-broadened lines, *Reviews of Modern Physics*, *17*, 227–236, 1945.

Wiscombe, W., The delta-m method: rapid yet accurate radiative flux calculations for strongly- asymmetric phase functions, *Journal of the Atmospheric Sciences*, *34*, 1408–1422, 1977.

Wolfe, W. L., *Introduction to Radiometry*, SPIE Optical Engineering Press, 1998.

Index