### TABLE A.1

<table>
<thead>
<tr>
<th>Physical Constants</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avogadro’s number</td>
<td>$N_a = 6.02214179 \times 10^{23}$ molecule mol$^{-1}$</td>
</tr>
<tr>
<td>Boltzmann’s constant</td>
<td>$k_B = 1.3806504$ J K$^{-1}$</td>
</tr>
<tr>
<td>Electron charge</td>
<td>$e = 1.602176487 \times 10^{-19}$ C</td>
</tr>
<tr>
<td>Gravitational constant</td>
<td>$G = 6.67429 \times 10^{-11}$ m$^3$ kg$^{-1}$ s$^{-2}$</td>
</tr>
<tr>
<td>Magnetic permeability of free space</td>
<td>$\mu_0 = 4\pi \times 10^{-7}$ kg m$^{-1}$ A$^{-2}$</td>
</tr>
<tr>
<td>Mass of an electron</td>
<td>$m_e = 9.10938215 \times 10^{-31}$ kg</td>
</tr>
<tr>
<td>Permittivity of free space</td>
<td>$\varepsilon_0 = 8.854187817 \times 10^{-7}$ kg$^{-1}$ m$^{-3}$ s$^{4}$ A$^{2}$</td>
</tr>
<tr>
<td>Planck’s constant</td>
<td>$h = 6.62606896 \times 10^{-34}$ J s</td>
</tr>
<tr>
<td>Stefan-Boltzmann constant</td>
<td>$\sigma = 5.670400 \times 10^{8}$ J m$^{-2}$ s$^{-1}$ K$^{-4}$</td>
</tr>
<tr>
<td>Speed of light in a vacuum</td>
<td>$c = 2.99792458 \times 10^{8}$ m s$^{-1}$</td>
</tr>
<tr>
<td>Universal gas constant</td>
<td>$R = 8.314472$ J mol$^{-1}$ K$^{-1}$</td>
</tr>
</tbody>
</table>
TABLE A.2
Astronomical Values

<table>
<thead>
<tr>
<th></th>
<th>g</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acceleration of gravity</td>
<td>9.80616 m s$^{-2}$</td>
<td>9.80665 m s$^{-2}$</td>
</tr>
<tr>
<td>(at sea level and 45° latitude)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Standard gravity</td>
<td>9.80616 m s$^{-2}$</td>
<td>9.80665 m s$^{-2}$</td>
</tr>
<tr>
<td>(nominal global average)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Angular velocity of rotation of the Earth</td>
<td>$7.27221 \times 10^5$ rad s$^{-1}$</td>
<td>$7.27221 \times 10^5$ rad s$^{-1}$</td>
</tr>
<tr>
<td>Average distance, Sun to Earth</td>
<td>$D_S$</td>
<td>$1.496 \times 10^8$ km</td>
</tr>
<tr>
<td>Average distance, Earth to Moon</td>
<td>$D_M$</td>
<td>$3.84 \times 10^6$ km</td>
</tr>
<tr>
<td>Radius of the Earth</td>
<td>$R_E$</td>
<td>6371 km</td>
</tr>
<tr>
<td>(volumetric mean)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average radius of the Moon</td>
<td>$R_M$</td>
<td>1740 km</td>
</tr>
<tr>
<td>Average radius of the Sun (visible disk)</td>
<td>$R_S$</td>
<td>$6.96 \times 10^8$ km</td>
</tr>
<tr>
<td>Average solar flux at TOA</td>
<td>$E_0$</td>
<td>1366 W m$^{-2}$</td>
</tr>
<tr>
<td>Mass of the earth</td>
<td>$M_E$</td>
<td>$5.988 \times 10^{24}$ kg</td>
</tr>
</tbody>
</table>

TABLE A.3
Relevant Meteorological Constants

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Density of air</td>
<td>$\rho$</td>
<td>$1.273 \times 10^{-3}$ g cm$^{-3}$; $1.273$ kg m$^{-3}$</td>
</tr>
<tr>
<td>at standard pressure and temperature</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Density of ice (0°C)</td>
<td>$\rho_i$</td>
<td>$0.917 \times 10^3$ kg m$^{-3}$</td>
</tr>
<tr>
<td>Density of liquid water (4°C)</td>
<td>$\rho_l$</td>
<td>$1 \times 10^3$ kg m$^{-3}$</td>
</tr>
<tr>
<td>Dry air gas constant</td>
<td>$R_{air}$</td>
<td>287 J kg$^{-1}$K$^{-1}$</td>
</tr>
<tr>
<td>Latent heat of fusion, ice</td>
<td>$L_{ice}$</td>
<td>$3.34 \times 10^3$ J kg$^{-1}$</td>
</tr>
<tr>
<td>Latent heat of vaporization, water at 0 °C</td>
<td>$L_v$</td>
<td>$2.50 \times 10^6$ J kg$^{-1}$</td>
</tr>
<tr>
<td>Molecular weight of dry air</td>
<td>$M$</td>
<td>28.97 g mol$^{-1}$</td>
</tr>
<tr>
<td>Saturation vapour pressure (0 °C)</td>
<td>$e_0$</td>
<td>6.1078 mb</td>
</tr>
<tr>
<td>Specific heat of air at constant pressure</td>
<td>$C_p$</td>
<td>$10.4 \times 10^2$ J kg$^{-1}$K$^{-1}$</td>
</tr>
<tr>
<td>Specific heat of air at constant volume</td>
<td>$C_v$</td>
<td>$7.17 \times 10^2$ J kg$^{-1}$K$^{-1}$</td>
</tr>
<tr>
<td>Standard pressure</td>
<td>$p_0$</td>
<td>1013.25 mb</td>
</tr>
<tr>
<td>Standard temperature</td>
<td>$T_0$</td>
<td>$273.16$ K</td>
</tr>
</tbody>
</table>
B

Mathematical Definitions and Identities

B.1 Mathematical Operators in Cartesian Coordinates

del

\( \nabla f = \frac{\partial f_x}{\partial x} \mathbf{i}_x + \frac{\partial f_y}{\partial y} \mathbf{i}_y + \frac{\partial f_z}{\partial z} \mathbf{i}_z \) \hfill (B.1)

divergence

\[ \text{div} f = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \]

\( = \nabla \cdot f \) \hfill (B.2)

curl Laplacian

\[ \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \] \hfill (B.3)

B.2 Matrix Algebra

For matrices A and B

\( (AB)^T = B^T A^T \) \hfill (B.5)

For a symmetric matrix S

\( \frac{\partial}{\partial x} (x - x_a)^T S^{-1} (x - x_a) = 2S(x - x_a) \) \hfill (B.6)

\( \frac{\partial}{\partial x} (y - Kx)^T S^{-1} (y - Kx) = -2K^T S(y - Kx) \) \hfill (B.7)
B.3 Legendre Polynomials

The Legendre polynomials $P_l(x)$, $l = 0, 1, 2, \ldots, $ are solutions of Legendre’s differential equation

$$ (1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + l(l + 1)y = 0 \quad (B.8) $$

when $x$ is a real number in the range $[-1, 1]$. For $l \geq 0$ the polynomial can be written

$$ P_l(x) = \sum_{r=0}^{m} \frac{(2l - 2r)!}{2^r r!(l - r)!(l - 2r)!} x^{l-2r} \quad (B.9) $$

where the integer $m$ is $l/2$ or $(l - 1)/2$. The explicit forms of the first six Legendre polynomials are

$$ P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x), \quad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \quad P_5(x) = \frac{1}{8}(65x^5 - 70x^3 + 15x). \quad (B.10) $$

The polynomials can be evaluated at the specific arguments $-1$, $0$ and $1$ to be

$$ P_l(-1) = (-1)^l \quad (B.11) $$
$$ P_l(0) = \frac{\cos(l\pi/2)\Gamma([l + 1]/2)}{\sqrt{\pi} \Gamma(l/2 + 1)} \quad (B.12) $$
$$ P_l(1) = 1 \quad (B.13) $$

Recurrence relations can be used to evaluate higher order Legendre polynomials. Formulae include

$$ P_{l+1}(x) = (2l + 1)P_l(x) + P_{l-1}(x) \quad (B.14) $$
$$ (l + 1)P_{l+1}(x) = (2l + 1)XP_l(x) - lP_{l-1}(x) \quad (B.15) $$
$$ P'_{l+1}(x) = XP'_l(x) + (l + 1)P_l(x) \quad (B.16) $$
$$ (x^2 - 1)P'_l(x) = xP_l(x) - lP_{l-1}(x) \quad (B.17) $$
$$ P''_l(x) = \frac{l(l+1)}{2l+1} \frac{1}{1-x^2} [P_{l+1}(x) - P_{l-1}(x)] \quad (B.18) $$

where the prime denotes differentiation with respect to $x$. Finally the Legendre polynomials are interrelated through

$$ P_{l-x} = (-1)^l P_l(x) \quad (B.19) $$
$$ P_{l-1}(x) = P_l(x) \quad (B.20) $$
B.4 Associated Legendre Polynomials

The equation for the associated Legendre polynomials is [Abramowitz and Stegun, 1964]
\[
(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \left[ l(l+1) - \frac{m^2}{1-x^2} \right] y = 0 \quad \text{(B.21)}
\]
has a solution (the solution needs a reference, also need to discuss arguments greater than 1)
\[
P_m^l(x) = (1 - x^2)^{m/2} \frac{d^m P_l(x)}{dx^m} \quad \text{(B.22)}
\]
where $x$ is restricted to $[-1, 1]$. Some authors include a factor $(-1)^m$ in the definition of $P_m^l(x)$. The associated Legendre polynomials have the following explicit forms
\[
P_0^0(x) = 1, \quad P_1^0(x) = x, \quad P_2^0(x) = \frac{1}{2} (3x^2 - 1), \quad P_2^1(x) = 3x(1 - x^2)^{1/2}
\]
\[
P_3^0(x) = \frac{1}{2} (5x^3 - 3x), \quad P_3^1(x) = \frac{1}{2} (5x^3 - 1)(1 - x^2)^{1/2}, \quad P_3^2(x) = 15(1 - x^2)
\]
also
\[
P_m^l(x) = P_l(x) \quad \text{and} \quad P_{l}^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_m^l(x). \quad \text{(B.24)}
\]
The associated Legendre polynomials obey the following recurrence relations
\[
(2l+1)P_{l+1}^m(x) = (l+m)P_l^m(x) + (l-m+1)P_{l-1}^m(x), \quad \text{(B.25)}
\]
\[
(1 - x^2)^{1/2} \frac{dP_l^m(x)}{dx} = \frac{1}{2} (l+m)(l-m+1)P_{l-1}^m(x) + \frac{1}{2} P_{l+1}^m(x), \quad \text{(B.26)}
\]
\[
(x^2-1)P_l^m(x) = l^2 P_l^m(x) - (l+m)P_{l-1}^m(x) \quad \text{(B.27)}
\]

B.5 Mie Angular Functions

The Mie angular functions $\pi_n$ and $\tau_n$ are defined in terms of the associated Legendre polynomials as
\[
\pi_n(\cos \Theta) = \frac{1}{\sin \Theta} P_n^l(\cos \Theta) \quad \text{(B.28)}
\]
\[
\tau_n(\cos \Theta) = \frac{d}{d\Theta} P_n^l(\cos \Theta). \quad \text{(B.29)}
\]
Evaluating the first terms gives
\[
\pi_0(\cos \Theta) = 0, \quad \pi_1(\cos \Theta) = 1, \quad \pi_2(\cos \Theta) = 3 \cos \Theta \quad \text{(B.30)}
\]
\[
\tau_0(\cos \Theta) = 0, \quad \tau_1(\cos \Theta) = \cos \Theta, \quad \tau_2(\cos \Theta) = 3 \cos 2\Theta
\]
\[
\pi_n(\cos \Theta) = \frac{1}{\sin \Theta} \left( \frac{(2n-1) \cos \Theta P_{n-1}(\cos \Theta) - (n-1)P_{n-2}(\cos \Theta)}{n} \right) \quad \text{(B.31)}
\]

\[
\frac{1}{\sin \Theta} \left( \frac{(2n-1) \cos \Theta P_{n-1}(\cos \Theta)}{n} \right) = \frac{(n-1)P_{n-2}(\cos \Theta)}{n} \quad \text{(B.32)}
\]

## B.6 Bessel Functions

The solutions to Bessel’s Equation
\[
x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 \quad \text{(B.33)}
\]
are called Bessel functions. The order of the Bessel function solution is determined by the constant \( n \). When \( n \) is not an integer the two solutions of Equation B.33 are the Bessel function of the first kind of order \( n \), \( J_n(x) \). Expressed as a series [see Boas, 2006, for the derivation] they are
\[
J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+1+n)} \left( \frac{x}{2} \right)^{2m+n} \quad \text{(B.34)}
\]
and
\[
J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+1-n)} \left( \frac{x}{2} \right)^{2n-2m} \quad \text{(B.35)}
\]
giving the general solution to Bessel’s equation of
\[
y(x) = c_1 J_n(x) + c_2 J_{-n}(x) \quad n \text{ not an integer} \quad \text{(B.36)}
\]
where \( c_1 \) and \( c_2 \) are constants.

To provide solutions at integer orders a combination of \( J_n(x) \) and \( J_{-n}(x) \) are used as the second solution of Bessel’s equation. These are \( Y_n(x) \) called Bessel functions of the second kind of order \( n \) and defined
\[
Y_n(x) = \frac{\cos \pi n J_n(x) - J_{-n}(x)}{\sin \pi n}. \quad \text{(B.37)}
\]

In some texts Bessel functions of the second kind are called the Weber functions or the Neumann functions (and so written \( N_n(x) \)). The general solution of Bessel’s equation applicable for all \( n \) is then
\[
y(x) = c_3 J_n(x) + c_4 Y_n(x), \quad \text{all } n \quad \text{(B.38)}
\]
where \( c_3 \) and \( c_4 \) are constants.
Appendix B: Mathematical Definitions and Identities

Finally a complex linear combination of Bessel functions of the first and second kinds is used to form Bessel functions of the third kind (also known as Hankel functions) $H_n^{(1)}$ and $H_n^{(2)}$. These are defined

\[
H_n^{(1)}(x) = J_n(x) + iY_n(x), \quad \text{and} \quad H_n^{(2)}(x) = J_n(x) - iY_n(x).
\]

Boas [2006] develops the following relations between the Bessel functions and their derivatives (which are also valid for the Neumann function),

\[
\frac{d}{dx}[x^nJ_n(x)] = x^nJ_{n-1}(x),
\]

\[
\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x),
\]

\[
J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x),
\]

\[
J_{n-1}(x) - J_{n+1}(x) = 2 J'_n(x),
\]

\[
J'_n(x) = -\frac{n}{x} J_n(x) + J_{n-1}(x) = \frac{n}{x} J_n(x) - J_{n+1}(x).
\]

B.7 Spherical Bessel Functions

The spherical Bessel function are defined in terms the Bessel functions of half-odd integer order. They are [Boas, 2006]

\[
j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x),
\]

\[
y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+1/2}(x),
\]

\[
h_n^1(x) = j_n(x) + iy_n(x) = \sqrt{\frac{\pi}{2x}} H^{(1)}_{n+1/2}(x),
\]

\[
h_n^2(x) = j_n(x) - iy_n(x) = \sqrt{\frac{\pi}{2x}} H^{(2)}_{n+1/2}(x).
\]

The power series expansions for the first and second order spherical Bessel functions are

\[
j_n(x) = 2^n x^n \sum_{m=0}^{\infty} \frac{(-1)^m n!}{m!(2n+2m+1)!} x^{2m},
\]

\[
y_n(x) = -\frac{1}{2^{n+1} x^{n+1}} \sum_{m=0}^{\infty} \frac{\Gamma(2n-2m+1)}{m! \Gamma(n-m+1)} x^{2m}.
\]
Using these expressions the Ricatti-Bessel functions for the first two orders are
\[ j_0(x) = 1 - \frac{x^2}{6} + \frac{x^4}{120}, \quad j_1(x) = \frac{x}{3} - \frac{x^3}{36} + \frac{x^5}{720}, \quad j_2(x) = \frac{x^2}{15} - \frac{x^4}{216} + \frac{x^6}{7200}, \quad j_0(x) = -\frac{1}{2} + \frac{x}{2} - \frac{x^3}{24}, \quad j_1(x) = -\frac{1}{2} - \frac{x}{2} - \frac{x^3}{24}. \] (B.52)

**B.8 Ricatti-Bessel Functions**

The Ricatti-Bessel functions \( \psi_n(x) \), \( \chi_n(x) \) and \( \zeta_n(x) \) are the product of the relevant spherical Bessel function with its argument, i.e.

\[ \psi_n(x) = x j_n(x) = \sqrt{\frac{x}{2}} J_{n+1/2}(x), \] (B.53)
\[ \chi_n(x) = x y_n(x) = \sqrt{\frac{x}{2}} Y_{n+1/2}(x), \] (B.54)
\[ \zeta_n(x) = \psi_n(x) + i \chi_n(x) = \sqrt{\frac{x}{2}} H^{(1)}_{n+1/2}(x). \] (B.55)

The infinite series expressions are

\[ \psi_n(x) = 2^n x^{n+1} \sum_{m=0}^{\infty} \frac{(-1)^m(n+m)!}{m!(2n + 2m + 1)!} x^{2m}, \] (B.56)
\[ \chi_n(x) = -\frac{1}{2^n x^n} \sum_{m=0}^{\infty} \frac{\Gamma(2n - 2m + 1)}{m!\Gamma(n - m + 1)} x^{2m}. \] (B.57)

Alternatively they can be written as a terminating series [Gumprecht and Sliepcevich, 1951]

\[ \psi_n(x) = \sin \left( x - \frac{nx}{2} \right) \sum_{m=0}^{\infty} \frac{(-1)^m(n+m)!}{(2m)!(n-2m)!(2x)^{2m}}, \]
\[ + \cos \left( x - \frac{nx}{2} \right) \sum_{m=0}^{\infty} \frac{(-1)^m(n+m)!}{(2m+1)!(n-2m-1)!(2x)^{2m+1}}, \] (B.58)
\[ \chi_n(x) = (-1)^n \cos \left( x + \frac{nx}{2} \right) \sum_{m=0}^{\infty} \frac{(-1)^m(n+m)!}{(2m)!(n-2m)!(2x)^{2m}}, \]
\[ - \sin \left( x + \frac{nx}{2} \right) \sum_{m=0}^{\infty} \frac{(-1)^m(n+m+1)!}{(2m+1)!(n-2m-1)!(2x)^{2m+1}}. \] (B.59)

Using these expressions the Ricatti-Bessel functions for the first two orders are

\[ \psi_0(x) = \sin x, \quad \psi_1(x) = \cos x, \quad \zeta_0(x) = \sin x + i \cos x, \]
\[ \psi_1(x) = \cos x, \quad \chi_1(x) = -\sin x, \quad \zeta_1(x) = \cos x - i \sin x \] (B.60)
Appendix B: Mathematical Definitions and Identities

Higher order values can be calculated using the recurrence relation

\[ \zeta_n(x) = \frac{2n - 1}{x} \zeta_{n-1}(x) - \zeta_{n-2}(x) \]  

(B.61)
C

Series Expansion of the Solution of the Spherical Wave Equation

The three dimensional wave or Helmholtz equation

\[ \nabla^2 \Pi + k^2 \Pi = 0 \quad (C.1) \]

can be expressed in spherical coordinates as [Kerker, 1969]

\[ \frac{1}{r} \frac{\partial^2 (r \Pi)}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Pi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Pi}{\partial \phi^2} + k^2 \Pi = 0 \quad (C.2) \]

Using the method of separation of variables, a solution of the form

\[ \Pi = R(r) \Theta(\theta) \Phi(\phi) \quad (C.3) \]

is adopted. Substituting this into Equation C.2 and multiplying by \( r^2 / R \Theta \Phi \) gives

\[ \frac{r}{R} \frac{\partial^2 (rR)}{\partial r^2} + \frac{1}{\Theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) + \frac{1}{\Phi \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} + r^2 k^2 \sin^2 \theta = 0 \quad (C.4) \]

which can be multiplied by \( \sin^2 \theta \) so that the third term is purely a function of \( \phi \). Separating the equation gives

\[ \frac{r}{R} \frac{\partial^2 (rR)}{\partial r^2} + \frac{1}{\Theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) + r^2 k^2 \sin^2 \theta = m^2, \quad (C.5) \]

\[ \frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial \phi^2} = -m^2. \quad (C.6) \]

The requirement of a single value solution implies that the solution for \( \Phi \) must have the same value for \( \phi + 2m \pi \) where \( m \) is an integer. Applying this restriction means that the solution for Equation C.6 is

\[ \Phi = a_m \cos (m \phi) + b_m \sin (m \phi). \quad (C.7) \]

where \( a_m \) and \( b_m \) are constants.

Equation C.5 can be rewritten as

\[ \frac{r}{R} \frac{\partial^2 (rR)}{\partial r^2} + \frac{1}{\Theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) + r^2 k^2 - \frac{m^2}{\sin^2 \theta} = 0 \quad (C.8) \]
so that by introducing a further constant \( p \), it can be separated into

\[
\frac{r}{R} \frac{\partial^2 (rR)}{\partial r^2} + r^2 k^2 = p
\]

\[
\frac{1}{\Theta \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) - \frac{m^2}{\sin^2 \theta} = -p
\]

Substituting \( x = \cos \theta \) reveals that Equation C.10 is the equation for the associated Legendre functions [Abramowitz and Stegun, 1964]

\[
(1 - x^2) \frac{d^2 \Theta}{dx^2} - 2x \frac{d \Theta}{dx} + \left[ l(l + 1) - \frac{m^2}{1 - x^2} \right] \Theta = 0
\]

given that \( p = l(l + 1) \) and \( l \) must be an integer in order that the solutions are finite at \( x = \cos \theta = \pm 1 \). The solution of C.10 is then

\[
\Theta = P^m_l(x)
\]

where \( P^m_l \) are the associated Legendre functions. They can be evaluated by means of recursion relationships defined in Appendix B.

The radial expression (Equation C.9) is now expressed as

\[
\frac{\partial^2 (rR)}{\partial r^2} + \left[ k^2 - \frac{l(l + 1)}{r^2} \right] rR = 0
\]

and solved by making the substitutions

\[
kr = \rho \quad \text{and} \quad R(r) = \frac{1}{\sqrt{\rho}} Z(\rho)
\]

to obtain the Bessel equation of half integer order

\[
\rho^2 \frac{d^2 Z}{d\rho^2} + \rho \frac{dZ}{d\rho} + \left[ \rho^2 - \left( l + \frac{1}{2} \right)^2 \right] Z = 0
\]

The two solutions of this expression are the half integral order Bessel function \( J_{l+1/2}(\rho) \) and the half integral order Neumann function \( N_{l+1/2}(\rho) \). The general solution of Bessel’s equation [Boas, 2006] may be written as

\[
Z(\rho) = AJ_{l+1/2}(\rho) + BN_{l+1/2}(\rho)
\]

where \( A \) and \( B \) are arbitrary constants. The solution of Equation C.9 to be written as

\[
R(r) = \frac{1}{kr} \left[ AJ_{l+1/2}(kr) + BN_{l+1/2}(kr) \right]
\]

Choosing different constants \( c_l \) and \( d_l \) defined by

\[
c_l = \frac{A}{k} \sqrt{\frac{2}{\pi}} \quad \text{and} \quad d_l = -\frac{B}{k} \sqrt{\frac{2}{\pi}}
\]
Appendix C: Series Expansion Solution of the Spherical Wave Equation

allows Equation C.17 to be expressed as

\[ rR(r) = c_l \sqrt{\frac{\pi kr}{2}} J_{l+1/2}(kr) - d_l \sqrt{\frac{\pi kr}{2}} N_{l+1/2}(kr) \]  
\[ = c_l \psi_l(kr) + d_l \chi_l(kr) \]  

where \( \psi_l(\rho) \) and \( \chi_l(\rho) \) are the Ricatti-Bessel functions which are defined in terms of the Bessel and Neumann functions [Kerker, 1969] by

\[ \psi_l(\rho) = \sqrt{\frac{\rho}{2}} J_{l+1/2}(\rho), \]  
\[ \chi_l(\rho) = -\sqrt{\frac{\rho}{2}} N_{l+1/2}(\rho). \]  

Hence

\[ rR = c_l \psi_l(kr) + d_l \chi_l(kr) \]  

The Ricatti-Bessel functions can be evaluated by means of recursion relationships defined in Appendix B. The functions \( \chi_l(kr) \) become infinite at the origin so can not be used to represent a wave where \( r \) can equal 0.

The general solution of the wave equation in spherical coordinates is obtained from a linear superposition of all the particular solutions

\[ r \Pi = r \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \pi_n^m \]
\[ = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[ c_l \psi_l(kr) + d_l \chi_l(kr) \right] P_l^m(\cos \theta) \left[ a_m \cos (m\phi) + b_m \sin (m\phi) \right] \]

\[ = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[ c_l \psi_l(kr) + d_l \chi_l(kr) \right] P_l^m(\cos \theta) \left[ a_m \cos (m\phi) + b_m \sin (m\phi) \right] \]  

(C.24)
References
References


Kattawar, G., S. J. Hitzfelder, and J. Binstock, An explicit form of the mie phase
References


Shannon, C., and W. Weaver, *The Mathematical Theory of Communication*, Univer-
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