The estimation of cloud droplet number concentration from satellite remote sensing data

A. C. Povey

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Abstract

A summary of calculations and assumptions used to derive expressions for cloud droplet number concentration (CDNC) and cloud physical thickness.

1 Definitions

Consider a cloud that extends[1] from a height $z = 0$ to H containing droplets that conform to a size distribution $n(r, z)$. The CDNC is defined as,

$$N_d(z) = \int_0^\infty n(r, z)\,\mathrm{d}r. \tag{1}$$

Based on in situ observations, this is assumed to be constant with height (neglecting the small regions of deviation near the edges of the cloud).

Satellite remote sensing determines the optical thickness of the cloud,

$$\tau_c = \int_0^H \beta_{ext}(z)\,\mathrm{d}z, \tag{2}$$

where the extinction of the droplets is,

$$\begin{aligned} \beta_{ext}(z) &= \int_0^\infty Q_{ext}(r) n(r) \pi r^2 \mathrm{d}r & (3)\\ &= \pi Q \int_0^\infty r^2 n(r)\,\mathrm{d}r & (4)\\ &\equiv \pi Q \langle r^2 \rangle, & (5) \end{aligned}$$

where Q_{ext} is the extinction efficiency of droplets, which has been assumed to have constant value $Q = 2$ because $r \gg \lambda$ (the wavelength of observation), and we introduced the notation,

$$\langle f \rangle = \int_0^\infty f n(r)\,\mathrm{d}r. \tag{6}$$

Satellite remote sensing also determines the effective radius of the cloud,

$$r_e = \langle r^3 \rangle / \langle r^2 \rangle. \tag{7}$$

Also relevant are the volume mean radius,

$$r_v = \sqrt[3]{\langle r^3 \rangle / N_d}, \tag{8}$$

and liquid water content,

$$l(z) = \tfrac{4}{3}\pi \rho_w \langle r^3 \rangle, \tag{9}$$

where ρ_w is the density of water.

2 Lapse rates

2.1 Dry air

The first law of thermodynamics states that,

$$\mathrm{d}Q = C_v \mathrm{d}T + p \mathrm{d}V, \tag{10}$$

where Q is the parcel's internal energy, C_v is its heat capacity at constant volume, T is temperature, p is pressure, and V is volume. An adiabatic change of the parcel is one that conserves the internal energy such that,

$$C_v \mathrm{d}T = -p \mathrm{d}V. \tag{11}$$

Differentiating the ideal gas law,

$$\begin{aligned} p\mathrm{d}V + V\mathrm{d}p &= M R_a\,\mathrm{d}T & (12)\\ V\mathrm{d}p &= (M R_a + C_v)\,\mathrm{d}T & (13)\\ &= C_p \mathrm{d}T & (14) \end{aligned}$$

[1]For simplicity, we will assume there is negligible scattering outside of the cloud.

where R_a is the specific gas constant of dry air, M is the mass of the parcel, and C_p is the heat capacity at constant pressure.

For an atmosphere in hydrostatic balance, the dry adiabatic lapse rate is then,

$$-\frac{\mathrm{d}T}{\mathrm{d}z} = -\frac{\mathrm{d}T}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}z} \tag{15}$$

$$\Gamma_d = -\frac{V}{C_p}(-\rho g) \tag{16}$$

$$= \frac{g}{c_p}, \tag{17}$$

where ρ is the density of air, g is gravitational acceleration, and the switch to a lower-case c indicates specific heat capacity.

2.2 Moist air

The saturation vapour pressure of water, e_s, can be calculated from the Clausius-Clapeyron equation, but several approximations have been proposed. Ed Gryspeerdt (personal communication, 2022) used the formulation of the Magnus equation proposed in Alduchov and Eskridge (1995),

$$e_s(T) = 610.97 \exp\left(\frac{17.625T}{T + 243.04}\right) \text{Pa}, \tag{18}$$

where T is in Celsius. In the same units, the latent heat of vaporization can be written,

$$L(T) = (2501 + 1.86T)\,\mathrm{kJ\,kg^{-1}}. \tag{19}$$

However, that paper describes (18) as the "saturation vapor pressure of pure water vapor over a plane surface of water". For "moist air above a plane surface of water", which sounds more appropriate to our circumstances, they recommend the adjustment,

$$e_{sa}(T, p) = 1.00071 \exp\left(4.5 \times 10^{-8} p\right) e_S(T). \tag{20}$$

The curved surface of the droplets also affects the vapour pressure according to Kelvin's equation,

$$e_{sc}(T, r) = e_s(T) \exp\left(\frac{2\sigma}{n_L RTr}\right), \tag{21}$$

where $\sigma \simeq 0.0756\,\mathrm{J\,m^{-2}}$ is the surface tension of water and n_L is the number of moles of liquid present. This is not yet included in the remainder of this proof.

Other approximations for e_s are available that may be more accurate for boundary layer clouds, as Alduchov and Eskridge (1995) concentrated on finding a forumla accurate over $[-40, 50°\mathrm{C}]$.

For a parcel saturated with water vapour, we approximate the equation of state as,

$$(p - e_s)V = MR_aT, \tag{22}$$

and make the pseudo-adiabatic approximation that the only change in internal energy is due to latent heat,

$$\mathrm{d}Q = -LM\mathrm{d}\mu_s, \tag{23}$$

where the saturation mass mixing ratio of water is,

$$\mu_s = \frac{\epsilon e_s(T)}{p - e_s(T)}, \tag{24}$$

where $\epsilon \simeq 0.622$ is the ratio of the molecular masses of water vapour and moist air.

Differentiating the log of that,

$$\frac{\mathrm{d}\mu_s}{\mu_s} = \frac{p}{p - e_s}\frac{\mathrm{d}e_s}{e_s} - \frac{\mathrm{d}p}{p - e_s} \tag{25}$$

$$= \frac{p}{p - e_s}\frac{L\mathrm{d}T}{R_vT^2} + \frac{g\,\mathrm{d}z}{R_aT}, \tag{26}$$

having substituted the first term on the RHS for the Clausius-Clapeyron equation (assuming vapour is substantially more voluminous than liquid) and the second term for hydrostatic balance of *dry* air. Note that $R_v = R_a/\epsilon$ is the specific gas constant for water vapour.

Substituting (26), (23) and (12) into (10) for a hydrostatic atmosphere,

$$0 = c_p\mathrm{d}T + g\mathrm{d}z + L\mathrm{d}\mu_s \tag{27}$$

$$= \left(c_p + \frac{p}{p - e_s}\frac{L^2\mu_s}{R_vT^2}\right)\mathrm{d}T + g\left(1 + \frac{L\mu_s}{R_aT}\right)\mathrm{d}z \tag{28}$$

Hence, the moist adiabatic lapse rate is,

$$-\frac{\mathrm{d}T}{\mathrm{d}z} = g\frac{1 + \frac{L\mu_s}{R_aT}}{c_p + \frac{p}{p-e_s}\frac{L^2\mu_s}{R_vT^2}} \tag{29}$$

$$\Gamma_m = \Gamma_d\left[1 + \frac{L\epsilon e_s}{R_aT(p - e_s)}\right] \tag{30}$$

$$\left[1 + \frac{L^2\epsilon^2 pe_s}{c_pR_aT^2(p - e_s)^2}\right]^{-1} \tag{31}$$

2.3 Vertical gradient of vapour

Dividing (27) by dz,

$$-L\frac{\mathrm{d}\mu_s}{\mathrm{d}z} = c_p\frac{\mathrm{d}T}{\mathrm{d}z} + g \tag{32}$$

$$-\frac{\mathrm{d}\mu_s}{\mathrm{d}z} = \frac{c_p}{L}\left(\Gamma_d - \Gamma_m\right). \tag{33}$$

3 CDNC

3.1 k-ratio

Assume that, within stratocumulus clouds, the ratio of r_e to r_v is approximately constant such that,

$$k = \left(\frac{r_v}{r_e}\right)^3 = \frac{\langle r^3\rangle}{N_d r_e^3} \tag{34}$$

$$\langle r^3\rangle = kN_d r_e^3. \tag{35}$$

This assumptions follows from the cloud droplets conforming to a gamma distribution $n(r) = N_0 r^{(1-3v_e)/v_e}\exp(-r/r_e v_e)$, for which $k = (1-v_e)(1-2v_e)$ where v_e is the effective variance of the distribution. Grosvenor et al. (2018) (hereafter G18) mentions values between 0.67 and 0.88 are used, though the ratio has been observed to vary with height in a manner inconsistent with the simple deposition process expected. MODIS and ORAC assume that $v_e = 0.111$ such that $k = 0.692$.

3.2 Sub-adiabatic lift

We then assume that the liquid water content is some constant fraction f_{ab} of its adiabatic value,

$$l(z) \equiv \tfrac{4}{3}\pi\rho_w\langle r^3\rangle = f_{ab}c_w z, \tag{36}$$

where c_w is the gradient of water mixing ratio with height.

G18 states that f_{ab} varies between 0.1 and 0.9, with adiabaticity suppressed by entrainment and precipitation. The value has been explored in situ and by systematic radar studies. It is widely assumed that $f_{ab} = 0.66$ or 0.67.

The condensation rate is derived from (33),

$$c_w = -\rho\frac{\mathrm{d}\mu_s}{\mathrm{d}z} = \frac{c_p(p-e_s)}{LR_aT}\left(\Gamma_d - \Gamma_m\right) \tag{37}$$

G18 asserts that ignoring the vertical variations in c_w produce errors of a few percent and, interestingly, implies that errors from using atmospheric pressure from reanalysis to represent that variation are larger than that.

3.3 Integration

Combining (35) and (36),

$$z = \frac{4\pi\rho_w kN_d}{3f_{ab}c_w}r_e^3. \tag{38}$$

Combining (2), (5), and (35),

$$\tau_c = \pi Q\int_0^H \langle r^2\rangle\,\mathrm{d}z \tag{39}$$

$$= \pi Q\int_0^H \frac{\langle r^2\rangle}{\langle r^3\rangle}\langle r^3\rangle\,\mathrm{d}z \tag{40}$$

$$= \pi Q\int_0^H kN_d r_e^2\,\mathrm{d}z. \tag{41}$$

Changing the variable of integration to r_e and noting that we have assumed everything other than it is constant through the depth of the cloud,

$$\tau_c = \pi Q\int_0^H kN_d r_e^2\frac{4\pi\rho_w kN_d}{f_{ab}c_w}r_e^2\,\mathrm{d}r_e \tag{42}$$

$$= \frac{Qk^2N_d^2 4\pi^2\rho_w}{f_{ab}c_w}\frac{r_e^5(H) - r_e^5(0)}{5} \tag{43}$$

$$= \frac{4\pi^2k^2N_d^2Q\rho_w r_e^5}{5f_{ab}c_w} \tag{44}$$

where we have assumed that $r_e(0) \ll r_e(H)$ and dropped its dependence on H as the satellite only provides effective radius near cloud top.

Rearranging,

$$N_d = \frac{1}{2\pi k}\sqrt{\frac{5f_{ab}c_w\tau_c}{Q_{ext}\rho_w r_e^5}}. \tag{45}$$

Quaas et al. (2006) implements this, citing Brenguier et al. (2000), as,

$$N_d = \alpha_0\tau_c^{0.5}r_e^{-2.5}, \tag{46}$$

where α follows from evaluating (45) at 850 hPa and 280 K with $f_{ab} = 1$,

$$e_s(280) = 610.94\exp\left[\frac{17.625\times 6.85}{6.85 + 243.04}\right]$$
$$= 990.4\,\mathrm{Pa}$$

$$\Gamma_d = \frac{9.81}{1.004\times10^3} = 9.771\times10^{-3}\,\mathrm{K\,m^{-1}}$$

$$\begin{aligned}\Gamma_m(280, 85000) &= 9.771\times10^{-3}\times \\ &\left[1+\frac{2.501\times10^6\times0.622\times990.4}{287.04\times280\times84010}\right] / \\ &\left[1+\left(\frac{2.501\times10^6\times0.622}{280\times84010}\right)^2 \right.\\ &\left.\frac{8.5\times10^5\times990.4}{1.004\times10^3\times287.04}\right] \\ &= 5.269\times10^{-3}\,\mathrm{K\,m^{-1}}\end{aligned}$$

$$\begin{aligned}c_w(280, 85000) &= \frac{1.004\times10^3\times84010}{2.501\times10^6\times287.04\times280} \\ &\times(9.771-5.269)\times10^{-3} \\ &= 1.889\times10^{-6}\,\mathrm{kg\,m^{-4}}\end{aligned}$$

$$\begin{aligned}\alpha_0(280, 85000) &= \frac{1}{2\pi\times0.80}\sqrt{\frac{5\times1\times1.889\times10^{-6}}{2\times10^3}} \\ &= 1.367\times10^{-5}\,\mathrm{m^{-1/2}}.\end{aligned}$$

3.4 Temperature dependence

Gryspeerdt et al. (2016) adds a temperature-dependent correction such that,

$$\alpha(T) = (0.0192T - 4.293)\,\alpha_0. \tag{47}$$

Personal communication with Ed revealed that it was calculated by assuming that, as temperature decreases, the saturation vapour pressure drops and the residual vapour condenses into droplets with gradient,

$$c_w(T) \simeq \frac{f_{ab}}{R_v T}\frac{\mathrm{d}e_s}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}z}, \tag{48}$$

but the value of k was mistakenly[2] used for f_{ab}. The derivative of e_s was calculated by finite difference and he used an approximate expression for Γ_m whereby $L(T)$ is constant and in (26) one assumes $e_s \ll p$. The final values were found using a linear fit to c_w in the range $[270, 300]$ in 1 K steps.

[2] Though 0.8 is a plausible value for f_{ab} is isn't the most common.

Alternatively, a linearisation of α may also be obtained via a Taylor expansion,

$$\begin{aligned}\alpha(T,p) \simeq \alpha(T_0,p_0) &+ (T-T_0)\left.\frac{\mathrm{d}\alpha}{\mathrm{d}T}\right|_{T=T_0,p_0} \\ &+ (p-p_0)\left.\frac{\mathrm{d}\alpha}{\mathrm{d}p}\right|_{T_0,p=p_0} + \ldots\end{aligned} \tag{49}$$

(45) can be written as,

$$\begin{aligned}N_d^2 = &\frac{5 f_{ab} g\epsilon}{4\pi^2 k^2 Q_{ext}\rho_w R_a}\frac{\tau_c}{r_e^5}\frac{e_s(p-e_s)}{T} \\ &\frac{\epsilon L p - (p-e_s)c_p T}{R_a c_p (p-e_s)^2 T^2 + \epsilon^2 L^2 p e_s}.\end{aligned} \tag{50}$$

The total derivative of the log of that is,

$$\begin{aligned}2\frac{\mathrm{d}N_d}{N_d} = &\mathrm{d}T\left[\frac{1}{T} + \frac{c_p(p-e_s)}{D_0} + \frac{2R_a c_p (p-e_s)^2 T}{D_1}\right] \\ &+\mathrm{d}e_s\left[\frac{1}{e_s} - \frac{1}{p-e_s} + \frac{c_p T}{D_0}\right. \\ &\left.+\frac{2R_a c_p (p-e_s) T^2 - \epsilon^2 L^2 p}{D_1}\right] \\ &+\mathrm{d}L\left[\frac{\epsilon p}{D_0} - \frac{2\epsilon^2 L p e_s}{D_1}\right] \\ &+\mathrm{d}p\left[\frac{1}{p-e_s} + \frac{\epsilon L - c_p T}{D_0}\right. \\ &\left.-\frac{2R_a c_p (p-e_s) T^2 + \epsilon^2 L^2 e_s}{D_1}\right],\end{aligned} \tag{51}$$

where $D_0 = \epsilon L p - (p-e_s)c_p T$, $D_1 = R_a c_p (p-e_s)^2 T^2 + \epsilon^2 L^2 p e_s$ and

$$\frac{\mathrm{d}e_s}{\mathrm{d}T} = \frac{BCe_s}{(T-273.15+C)^2}, \tag{52}$$

where B, C are the constants in the numerator and denominator of (18), respectively.

For the same assumptions as (47), namely $f_{ab} = 1$ and $\frac{\mathrm{d}L}{\mathrm{d}T} = 0$, linearising the above around 275 K and 850 hPa gives,

$$\begin{aligned}\alpha(T) = &(0.0145T + 2.817\times10^{-6}p - 3.2314) \\ &\times 1.282\times10^{-5}\,\mathrm{m^{-1/2}}.\end{aligned} \tag{53}$$

This confirms the frequent assertion that α varies weakly with pressure.

Using $f_{ab} = 0.66$, $k = 0.692$, (19) and (20)

$$\alpha_{ORAC}(T) = (0.0145T - 3.001) \times 1.205 \times 10^{-5}\,\mathrm{m}^{-1/2}. \tag{54}$$

3.5 Thickness

The physical thickness of the cloud can be derived by using $l(H)$ to replace $\pi k N_d$ in (44),

$$\tau_c = \frac{4Q\rho_w r_e^5}{5 f_{ab} c_w}\left(\frac{3 f_{ab} c_w H}{4\rho_w r_e^3}\right)^2 \tag{55}$$

$$= \frac{9Q f_{ab} c_w H^2}{20\rho_w r_e} \tag{56}$$

$$H = \sqrt{\frac{20\rho_w \tau_c r_e}{9Q f_{ab} c_w}}. \tag{57}$$

This is (4) of Meerkötter and Zinner (2007), with $Q_{ext} = 2$ and $f_{ab} = 1$.

Alternatively, one can find the liquid water path,

$$\mathrm{LWP} = \int_0^H l(z)\,\mathrm{d}z \tag{58}$$

$$= \int_0^H f_{ab} c_w z\,\mathrm{d}z \tag{59}$$

$$= \tfrac{1}{2} f_{ab} c_w H^2 \tag{60}$$

$$H = \sqrt{\frac{2\mathrm{LWP}}{f_{ab} c_w}}. \tag{61}$$

Comparing the two expression, we arrive at the sub-adiabatic expression for liquid water path,

$$\mathrm{LWP} = \frac{10}{9Q}\rho_w \tau_c r_e. \tag{62}$$

If r_e is instead assumed to be constant with height, the homogenous expression for liquid water path is recovered,

$$\mathrm{LWP} = \int_0^H \frac{4}{3Q}\rho_w r_e \beta_{ext}\,\mathrm{d}z = \frac{4}{3Q}\rho_w \tau_c r_e. \tag{63}$$

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