3

Radiometric Basics

The representation of a light beam as an electromagnetic wave or a collection of photons is very useful in understanding the way light interacts with matter. However, in describing light in the atmosphere we are principally concerned with the flow of energy. This section introduces the terms and concepts used in such a radiometric description.

3.1 Solid Angle

A plane angle (α) is the ratio of the length of an arc (c) to the radius of the arc (r). In an analogous way the solid angle ω is the ratio of an area A to the square of the radius of a sphere. The shape of the area does not matter. This is shown in Figure 3.1. Hence solid angle is a measure of the angular size in two dimensions of an object.

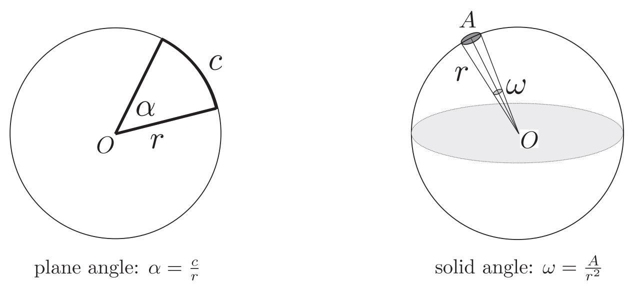

FIGURE 3.1
Definition of plane angle and solid angle.

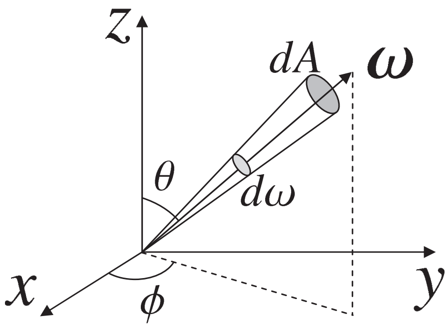

FIGURE 3.2
Definition of azimuth angle, ϕ, zenith angle, θ, (this direction pair is abbreviated as $\boldsymbol{\omega}$) and differential solid angle $d\omega$.

Figure 3.2 defines the azimuth angle, ϕ, and zenith angle, θ, used to specify direction. In many cases the flow of energy is found by summing the contribution of many rays crossing a surface travelling in different directions. Therefore it is worth noting the standard integral results found using this notation,

$$+z \text{ hemisphere} \qquad \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta\, d\phi = 2\pi \tag{3.1}$$

$$-z \text{ hemisphere} \qquad \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} \sin\theta\, d\theta\, d\phi = 2\pi \tag{3.2}$$

$$\text{sphere} \qquad \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi = 4\pi \tag{3.3}$$

$$\text{projected hemisphere} \qquad \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta\, d\theta\, d\phi = \pi \tag{3.4}$$

where the terms $+z$ hemisphere and $-z$ hemisphere have been introduced to describe the set of directions where θ is in the range $[0, \pi/2]$ or $[\pi/2, \pi]$ respectively. At times it is convenient to make the substitution $\mu = \cos\theta$ and use μ instead of θ as a coordinate. Alternatively, equations are simplified by abbreviating the direction pair (θ, ϕ) as the vector $\boldsymbol{\omega}$.

Now consider the area, dA, created by differential changes in the angular coordinates, i.e.

$$dA = r d\theta \times \sin\theta\, d\phi. \tag{3.5}$$

The differential solid angle, $d\omega$, is formed by dividing by r^2 i.e

$$d\omega = \frac{dA}{r^2} = \sin\theta d\theta\, d\phi. \tag{3.6}$$

Rather than integrate over θ and ϕ the differential solid angle can be used as a shorthand notation for integration over solid angle. So, for example, for some function $f(\theta, \phi)$, $f(\mu, \phi)$ or $f(\boldsymbol{\omega})$ equivalent integral expression are

$+z$ hemisphere integration

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} f(\theta,\phi)\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{1} f(\mu,\phi)\, d\mu\, d\phi = \int_0^{2\pi} f(\boldsymbol{\omega})\, d\omega, \quad (3.7)$$

$-z$ hemisphere integration

$$\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} f(\theta,\phi)\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_{-1}^{0} f(\mu,\phi)\, d\mu\, d\phi = \int_0^{-2\pi} f(\boldsymbol{\omega})\, d\omega, \quad (3.8)$$

spherical integration

$$\int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_{-1}^{1} f(\mu,\phi)\, d\mu\, d\phi = \int_0^{4\pi} f(\boldsymbol{\omega})\, d\omega, \quad (3.9)$$

where the limits of the differential solid angle integral are given as 2π for the $+z$ hemisphere, -2π for the $-z$ hemisphere, and 4π for the integral over a sphere. When using this notation care should be taken to distinguish the direction vector $\boldsymbol{\omega}$ from the differential solid angle $d\omega$.

3.2 Radiant Quantities

The radiant energy, Q, is the energy emitted, transferred or received by electromagnetic waves. The flow of radiant energy is generally achieved by a large numbers of photons and described by four fundamental terms [*ISO*, 1992]

- Radiant flux, Φ,
- Irradiance, E,
- Radiant intensity I,
- Radiance, L.

The French led the naming and development of radiometric standards so symbols for the last three quantities, which seem anomalous to the English speaker, are derived from *eclairage* (E), *intensité* (I) and *luminosité* (L). Before defining and discussing these terms it should be noted that:

- All wavelengths contribute to radiant energy but the radiant energy, and terms describing its transport, can also be specified over a defined range of wavelengths.
- The point or surface where radiometric quantity is evaluated may be real or it may be imagined.

3.2.1 Radiant flux, Φ

The flow of radiant energy is encapsulated in the concept of radiant flux, Φ, defined as the rate at which radiant energy is transferred from a surface or point to another point or surface, i.e.

$$\Phi = \frac{\partial Q}{\partial t}, \tag{3.10}$$

where t is time.

3.2.2 Radiant Flux Density, M or E

The radiant flux density is a flow of energy per unit area, i.e.

$$M \text{ or } E = \frac{\partial \Phi}{\partial A}. \tag{3.11}$$

The radiant flux density is separated into radiant exitance from a surface M, and irradiance (or incidence) onto a surface E. However this terminology is only loosely respected with irradiance used for both incident and leaving radiation. One reason is that the distinction is not particularly useful for an imaginary surface where the irradiance onto the surface is identical to the exitance from the reverse of the surface.

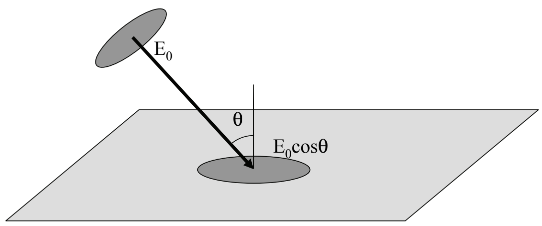

FIGURE 3.3
A collimated beam shining on a surface.

As an example of the use of irradiance to describe energy flow consider a collimated beam which near its axis can be considered a plane wave. If such a collimated beam is shining on a plane at an angle θ to the normal as shown in Figure 3.3 then the collimated beam illuminates an area $\sec\theta$ larger than the cross-section of the beam. Hence if the irradiance of the beam is E_0 (measured on the plane orthogonal to the direction of propagation) then the irradiance on the surface is $E_0 \cos\theta$.

3.2.3 Radiant intensity, I

Now consider the energy flow at two points d_1 and d_2 from a point light source generating a spherical electromagnetic wave as shown in Figure 3.4. If we consider

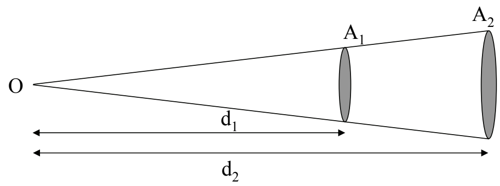

FIGURE 3.4
Element of a spherical wave generated at the origin, O. The two areas of interest A_1 and A_2 located at d_1 and d_2 respectively contain the same solid angle when observed from O.

an element of solid angle we know the rate of energy flow through each surface A_1 and A_2 is the same i.e.

$$\Phi_1 = \Phi_2. \tag{3.12}$$

However as the area over which this energy is spread changes the radiant flux density is no longer invariant. However the solid angle of each surface seen from the point source is the same. The invariant quantity is the radiant intensity, $I(\boldsymbol{\omega})$, defined as the radiant flux per unit solid angle travelling in direction $(\boldsymbol{\omega})$, i.e.

$$I(\boldsymbol{\omega}) = \frac{\partial \Phi}{\partial \omega}. \tag{3.13}$$

The average radiant intensity leaving a point, $\bar{I}$, is the radiated power divided by 4π, i.e.

$$\bar{I} = \frac{1}{4\pi} \int_0^{4\pi} I(\boldsymbol{\omega})\, d\omega. \tag{3.14}$$

3.2.4 Radiance, L

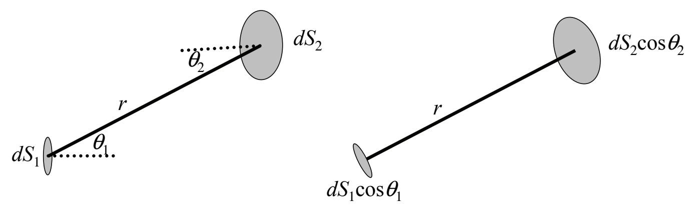

FIGURE 3.5
a) Two optical elements dS_1 and dS_2 separated by a distance r and having angles θ_1 and θ_2 between the normal to each area and the line joining the centres of the areas. b) When the areas are rotated to align their normals with the line joining their centres.

Generally light is not collimated nor does it originate from a point source. When the source has a measurable area it is termed an extended source . Consider two elemental areas dS_1 and dS_2 separated by a distance r and having angles θ_1 and θ_2 between the normal to each area and the line joining the centres of the areas. The medium between the areas is defined to be transparent. This construction is shown in Figure 3.5. The first area presents an effective area $dS_1 \cos\theta_1$ towards dS_2 while the second area has a solid angle of $dS_2 \cos\theta_2/r^2$ when viewed from S_1. Conversely dS_2 presents an effective area $dS_2 \cos\theta_2$ towards dS_1 while the first area has a solid angle of $dS_1 \cos\theta_1/r^2$ when viewed from S_2. The product of the effective area and solid angle from either component is the same i.e.

$$dG^2 = dS_1 \cos\theta_1 \frac{dS_2 \cos\theta_2}{r^2} = dS_2 \cos\theta_2 \frac{dS_1 \cos\theta_1}{r^2} \tag{3.15}$$

This term, dG^2 is called the étendue and represents the geometric linkage between two optical elements. If the intervening media involves a change in refractive index then a further n^2 term is included in the definition of étendue [e.g. see *Brooker*, 2003]. As the étendue is conserved through an optical system it is often called the throughput. In many optical system the surface normals are aligned with r so that the étendue is defined as the product of the surface area and the observed solid angle i.e.

$$dG^2_{\text{normally aligned}} = dS_1 \frac{dS_2}{r^2} = dS_2 \frac{dS_1}{r^2} \tag{3.16}$$

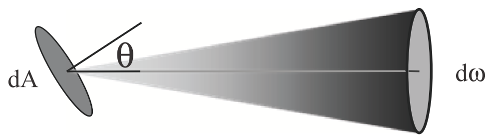

FIGURE 3.6
Radiance defined as the rate of energy leaving a surface element $dA\cos\theta$ in a direction θ (relative to the surface normal) per solid angle.

This concept of étendue leads to the definition of radiance, L, which is the rate of energy propagation in a given direction per unit solid angle per unit area perpendicular to the axis of the solid angle. This is shown in Figure 3.6. Radiance is the derivative of the radiative flux with respect to étendue

$$L = \frac{\partial^2\Phi}{\partial G^2} = \frac{\partial^2\Phi}{\partial\omega\,(\cos\theta\,\partial A)}, \tag{3.17}$$

where ∂A is the elemental area containing the point and $d\omega = \sin\theta\, d\theta d\phi$ is the differential solid angle centred about (θ, ϕ), the direction of travel of the ray. Note that a local coordinate system used to define radiance. In this system the z axis is aligned with the normal to the surface from which the ray is leaving. This ensures

θ is always in the range $[0, \pi/2]$. Note that when $\theta = \pi/2$ the ray carries no energy from the evaluating surface so the radiance is zero. In everyday terms the radiance can be interpreted as the brightness of an object. The product of the radiance and étendue is the power received by an optical system. As the eye can be approximated as an optical device with a fixed étendue, the power received by the eye is a linear function of observed radiance.

3.2.5 Spectral and Measurable Quantities

The term *spectral* is applied to radiometric quantities to denote spectral density. For instance the spectral radiant flux, $\Phi_\lambda(\lambda)$ is defined so that $\Phi_\lambda(\lambda)d\lambda$, is the radiant energy per unit time within the interval λ and $\lambda + d\lambda$. The spectral density can also be expressed in terms of wavenumber or frequency. The subscript λ, $\tilde{\nu}$ or ν is added to a term to denote the spectral density with respect to either wavelength, wavenumber or frequency. The radiant flux can be expressed in terms of the spectral radiant flux by

$$\Phi = \int_0^\infty \Phi_\lambda(\lambda)\,d\lambda = \int_0^\infty \Phi_{\tilde{\nu}}(\tilde{\nu})\,d\tilde{\nu} = \int_0^\infty \Phi_\nu(\nu)\,d\nu. \tag{3.18}$$

It should be realised that spectral quantities are infinitesimal and can never be measured. What can be measured is the energy within some range of wavelengths. For example a red filter restricts the energy observed to that with wavelengths between approximately 630 to 740 nm. Such a range is called a waveband or band and is defined by the upper and lower band limits, labelled here λ_1 and λ_2 respectively. The symbol $\Delta\lambda$, is used to indicate that a finite range of wavelengths is being considered. For example the waveband radiant flux density is given by

$$E(\Delta\lambda) = \int_{\lambda_1}^{\lambda_2} E_\lambda(\lambda)\,d\lambda \tag{3.19}$$

It is also not possible to measure any radiometric quantity that is a derivative respect to solid angle as all detectors have a finite size. Instead the qualifier *conical* is used to denote a radiant quantity integrated over a small but finite solid angle. Such a range is defined by the upper and lower band limits, labelled here $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ respectively. For example the waveband conical radiance is given by

$$L(\Delta\lambda, \Delta\omega) = \int_{\lambda_1}^{\lambda_2} \int_{\boldsymbol{\omega}_1}^{\boldsymbol{\omega}_2} L_\lambda(\lambda, \boldsymbol{\omega})\,d\omega\,d\lambda \tag{3.20}$$

Often the variation of the radiant quantity is negligible so that the integrals can be completed. For example

$$L(\Delta\lambda, \Delta\omega) = (\lambda_2 - \lambda_1)(\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1)L_\lambda(\lambda, \boldsymbol{\omega}) \tag{3.21}$$

where the value of spectral radiance comes from somewhere in the band and cone of interest.

TABLE 3.1
Radiometric Quantities

Name	Symbol	Common Unit
Radiant Energy	Q	J
Spectral Radiant Energy	Q_λ	J μm^{-1}
	$Q_{\tilde{\nu}}$	J (cm^{-1})$^{-1}$
Radiant Flux	Φ	W
Spectral Radiant Flux	Φ_λ	W μm^{-1}
	$\Phi_{\tilde{\nu}}$	W (cm^{-1})$^{-1}$
Radiant Flux Density		
Irradiance	E	W m^{-2}
Exitance	M	W m^{-2}
Spectral Radiant Flux Density		
Incidence/Irradiance	E_λ	W m^{-2} μm^{-1}
	$E_{\tilde{\nu}}$	W m^{-2} (cm^{-1})$^{-1}$
Exitance	M_λ	W m^{-2} μm^{-1}
	$M_{\tilde{\nu}}$	W m^{-2} (cm^{-1})$^{-1}$
Radiant Intensity	I	W sr^{-1}
Spectral Radiant Intensity	I_λ	W μm^{-1} sr^{-1}
	$I_{\tilde{\nu}}$	W (cm^{-1})$^{-1}$ sr^{-1}
Radiance	L	W m^{-2} sr^{-1}
Spectral Radiance	L_λ	W m^{-2} μm^{-1} sr^{-1}
	$L_{\tilde{\nu}}$	W m^{-2} (cm^{-1})$^{-1}$ sr^{-1}

Note that as wavenumbers are often expressed in cm^{-1} then the spectral dependence is usually shown as (cm^{-1})$^{-1}$.

Table 3.1 summarizes some of the radiometric quantities that have been introduced and units usually used. It is worth concluding with the caveat that the use of radiometric symbols between and within science disciplines is often contradictory. To quote *Wolfe* [1998]

> *Although radiometric terms can be used, misused, and abused in a number of ways, intensity may be the worst.*

To illustrate this example one just has to note that undergraduate optics texts typically use intensity to mean the flux density, while in astronomy it is the accepted term for what is defined here as radiance.

3.2.6 Relating Radiance and Irradiance

Consider a coordinate system with the z axis is aligned with the direction of propagation of ray of radiance L. The exitance from the $x - y$ plane is $E = L d\omega$. More generally if $x - y$ plane of coordinate system is defined by a surface element dA. As shown in Figure 3.7 the polar angle θ is defined with respect to the upward pointing normal from the elemental surface area dA of interest. The exitance from the surface

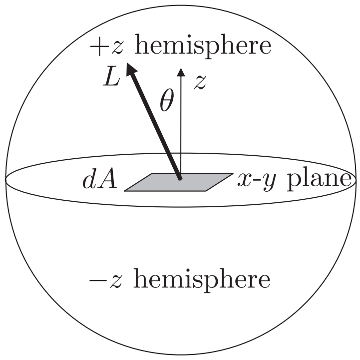

FIGURE 3.7
Geometry defined by a surface element *dA*.

element can be expressed in terms of the radiance of each incident ray by

$$dM = L(\boldsymbol{\omega})\ \cos\theta\, d\omega \tag{3.22}$$

The integral of all the rays where $\theta > \pi/2$ leaving the surface element gives the radiant exitance in the positive z direction $M(2\pi)$

$$M(+2\pi) = \int_0^{2\pi} L(\boldsymbol{\omega}) \cos\theta\, d\omega \tag{3.23}$$

This is shown graphically in Figure 3.8.

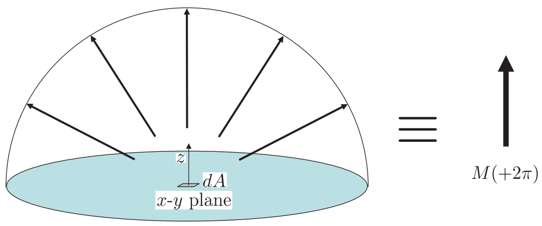

FIGURE 3.8
Graphical representation of the construction of radiant exitance from a radiance field.

A radiance field at some location is represented by $L(\boldsymbol{\omega})$ where $\boldsymbol{\omega}$ is the direction in which the ray is travelling. At this location we can define an equivalent radiant exitance from an elemental surface. The radiant flux density in the $+z$ direction (i.e. where $\theta = 0$), $\hat{M}$, is given by the spherical integral of the radiance leaving the surface

$$\hat{M} = \int_0^{2\pi} L(\boldsymbol{\omega}) \cos\theta\, d\omega. \tag{3.24}$$

The fact that this expression gives the exitance in the $+z$ direction and not the sum of the exitance leaving the surface from above and below may, at first instance, seem surprising. The reason is the $\cos\theta$ term changes sign for rays where $\theta > \pi/2$. Although useful in this context, the change of sign has an unfortunate side-effect when calculating the irradiance from the radiance field for rays where $\theta > \pi/2$. In this case the integral

$$\int_0^{2\pi} L(\boldsymbol{\omega}) \cos\theta \, d\omega \tag{3.25}$$

gives a negative answer i.e. the flow of energy is in the $-z$ direction. While mathematically exact a negative energy flow is not a helpful concept. To avoid this the differential projected solid angle $d\Omega$ is defined as

$$d\Omega = |\cos\theta| \, d\omega. \tag{3.26}$$

The absolute value of $\cos\theta$ prevents $d\Omega$ being negative for rays travelling in directions where θ lies in the range $\pi/2$ to π. Using this notation the radiant exitance in the $+z$ direction is

$$M(+2\pi) = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} L(\theta, \phi) \cos\theta \sin\theta \, d\theta \, d\phi = \int_0^{2\pi} L(\boldsymbol{\omega}) \, d\Omega. \tag{3.27}$$

whereas the radiant exitance in the $+z$ direction, $M(-2\pi)$, is

$$M(-2\pi) = \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} L(\theta, \phi) |\cos\theta| \sin\theta \, d\theta \, d\phi = \int_0^{-2\pi} L(\boldsymbol{\omega}) \, d\Omega. \tag{3.28}$$

For an imaginary surface the irradiance on one side of the surface must be identical to the exitance from the other side of the surface. This is shown in Figure 3.9 and can be represented mathematically as

$$E(+2\pi) \equiv M(+2\pi) \qquad \text{and} \qquad E(-2\pi) \equiv M(-2\pi) \tag{3.29}$$

where $E(+2\pi)$ and $E(-2\pi)$ represent the irradiance travelling in the $+z$ and $-z$ directions respectively. Using these equivalences gives the irradiance on the x-y plane in the $+z$ direction, as

$$E(+2\pi) = \int_0^{2\pi} L(\boldsymbol{\omega}) \, d\Omega. \tag{3.30}$$

Similarly the irradiance on the x-y plane in the $-z$ direction, as

$$E(-2\pi) = \int_0^{-2\pi} L(\boldsymbol{\omega}) \, d\Omega. \tag{3.31}$$

Now consider a collimated beam travelling in direction $\boldsymbol{\omega}$. The term $E(\boldsymbol{\omega})$ is used to represent the irradiance from a collimated beam incident on a plane orthogonal to $\boldsymbol{\omega}$. Using this notation the relationship between a collimated beam and the irradiance generated on the x-y plane is

$$\text{for a collimated beam:} \qquad \begin{array}{ll} E(+2\pi) = E(\boldsymbol{\omega}) \cos\theta & 0 < \theta < \pi/2 \\ E(-2\pi) = E(\boldsymbol{\omega}) |\cos\theta| & \pi/2 < \theta < \pi \end{array}. \tag{3.32}$$

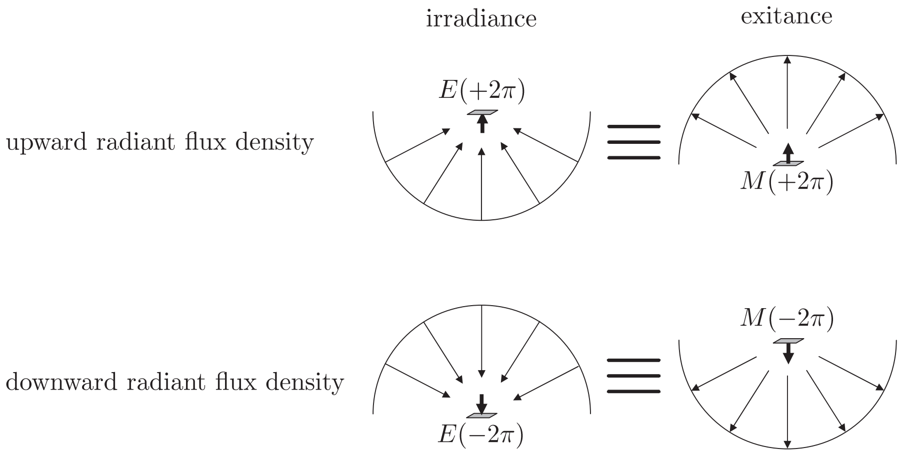

FIGURE 3.9
Graphical representation of the construction of the irradiance or exitance from a radiance field.

When the radiance is the same in all directions (i.e. isotropic) the integrations in Equations 3.30 and 3.31 can be performed, so the relationship between radiance and irradiance is

$$\text{isotropic only:} \qquad \begin{array}{ll} E(+\overline{2\pi}) = \pi L(\boldsymbol{\omega}) & 0 < \theta < \pi/2 \\ E(-\overline{2\pi}) = \pi L(\boldsymbol{\omega}) & \pi/2 < \theta < \pi \end{array} \tag{3.33}$$

where $E(+\overline{2\pi})$ and $E(-\overline{2\pi})$ have been used to show the irradiance is formed from a diffuse isotropic field. The irradiance terms that have been introduced are listed in Table 3.2.

TABLE 3.2
Directionality of Irradiance

Symbol	**Description**
$E(\boldsymbol{\omega})$	the irradiance caused by a unidirectional beam on the plane orthogonal to the $\boldsymbol{\omega}$ direction.
$E(+2\pi)$	the irradiance on the x-y plane in the $+z$ direction.
$E(-2\pi)$	the irradiance on the x-y plane in the $-z$ direction.
$E(+\overline{2\pi})$	the irradiance on the x-y plane in the $+z$ direction caused by an isotropic diffuse radiance field.
$E(-\overline{2\pi})$	the irradiance on the x-y plane in the $-z$ direction caused by an isotropic diffuse radiance field.

In the atmosphere the solar irradiance, $E^{\text{Sun}}(\boldsymbol{\omega}_0)$, is often treated as a unidirectional beam travelling in direction $\boldsymbol{\omega}_0 \equiv (\theta_0, \phi_0)$. This is equivalent to a radiance field described by

$$L(\boldsymbol{\omega}) = \delta(\boldsymbol{\omega} - \boldsymbol{\omega}_0) E^{\mathrm{Sun}}(\boldsymbol{\omega}_0) \tag{3.34}$$

where δ is a version of the Dirac delta function and has the following properties

$$\delta(\boldsymbol{\omega}) = \begin{cases} \infty & \boldsymbol{\omega} = 0 \\ 0 & \boldsymbol{\omega} \neq 0 \end{cases} \tag{3.35}$$

$$\int_0^{2\pi} \delta(\boldsymbol{\omega})\, d\omega = \int_0^{-2\pi} \delta(\boldsymbol{\omega})\, d\omega = 1 \tag{3.36}$$

Substituting Equation 3.34 into Equation 3.31 gives the expected result for the solar irradiance on a horizontal surface in the absence of an atmospheric which is

$$\int_0^{-2\pi} \delta(\boldsymbol{\omega} - \boldsymbol{\omega}_0) E^{\mathrm{Sun}}(\boldsymbol{\omega}_0)\, d\Omega = \cos\theta_0 E^{\mathrm{Sun}}(\boldsymbol{\omega}_0). \tag{3.37}$$

3.2.7 Relationships between Electromagnetic and Radiometric Quantities

The radiant flux density is a flow of energy per unit area, i.e.

$$M \text{ or } E = \frac{\partial \Phi}{\partial A} = \langle S \rangle. \tag{3.38}$$

The magnitude of the radiant flux density is the magnitude of the time averaged Poynting vector, $\langle S \rangle$, which was defined in Equation 2.98. For a plane harmonic wave the relationship is

$$E = \frac{1}{2} \epsilon_0 c E_0^2. \tag{3.39}$$

The power per unit steradian (i.e. the radiometric intensity) for a spherical harmonic wave is

$$I = \langle \mathbf{S} \rangle r^2 = \frac{1}{2} \epsilon_0 c E_0^2 d^2, \qquad [\mathrm{W\ sr^{-1}}] \tag{3.40}$$

where d is the distance from the point source.

The electromagnetic and radiometric descriptions of light can be connected through the concept of spectral energy density introduced in Section 2.5. Recall that the spectral energy density u_ν is the energy per unit volume in the frequency range from ν to $\nu + d\nu$. To relate the energy density to radiometric quantities we consider two cases:

For a collimated beam of cross-section A all the energy in a layer ct thick will cross A in time t. This can be equated to the radiant power by

$$\Phi_\nu t = u_\nu A c t \tag{3.41}$$

Dividing by the area gives the spectral irradiance in terms of spectral energy density

$$E_\nu = u_\nu c \tag{3.42}$$

If the radiation field is isotropic then the flow of radiant power is the same in all directions. The spectral radiance is found by multiplying the spectral energy density by the velocity of flow c and dividing by 4π, i.e.

$$L_\nu(\nu) = \frac{u_\nu \times c}{4\pi} \tag{3.43}$$

3.3 Blackbody Radiation

Electromagnetic radiation generated by the thermal motion of charged particles in matter. All matter with a temperature greater than absolute zero emits thermal radiation. A black body is an idealized physical body that absorbs all incident electromagnetic radiation. Because of this perfect absorptivity at all wavelengths, a black body is also the best possible emitter of thermal radiation.

3.3.1 Planck Function

The standard method for developing the spectral distribution of light emitted from a body is to consider a cubical cavity of side length l where the walls are at a fixed temperature and the radiation within the enclosure is in equilibrium.

The standing waves that are possible with in the cavity must have discrete wavelengths so that at the walls ($x = 0$ & $x = l$), the amplitude of the wave is zero. If the wave is normal to a wall then non-zero amplitudes can only occur for waves that have discrete wavelengths such that

$$\frac{n\lambda}{2} = l, \tag{3.44}$$

where $n = 1, 2, 3, \ldots$. If the wave is in some arbitrary direction whose direction normal makes angles α, β, γ with the x, y and z axes then the valid wavelengths are defined by

$$\frac{n_x\lambda}{2} = l\cos\alpha, \qquad \frac{n_y\lambda}{2} = l\cos\beta, \qquad \frac{n_z\lambda}{2} = l\cos\gamma, \tag{3.45}$$

where n_x, n_y and n_z define the number of half wavelengths in the x, y and z directions respectively. Squaring and summing these equations gives

$$n_x^2 + n_y^2 + n_z^2 = \left(\frac{2l}{\lambda}\right)^2 \left(\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right). \tag{3.46}$$

From Pythagoras $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ so

$$n_x^2 + n_y^2 + n_z^2 = \left(\frac{2l}{\lambda}\right)^2. \tag{3.47}$$

Rearranging this equation gives the permitted frequencies of the standing waves as

$$\nu = \sqrt{n_x^2 + n_y^2 + n_z^2}\,\frac{c}{2l}, \tag{3.48}$$

for all positive integers n_x, n_y and n_z. It is helpful to introduce the radial coordinate r defined by

$$r^2 = n_x^2 + n_y^2 + n_z^2, \tag{3.49}$$

so

$$\nu = \frac{rc}{2l}. \tag{3.50}$$

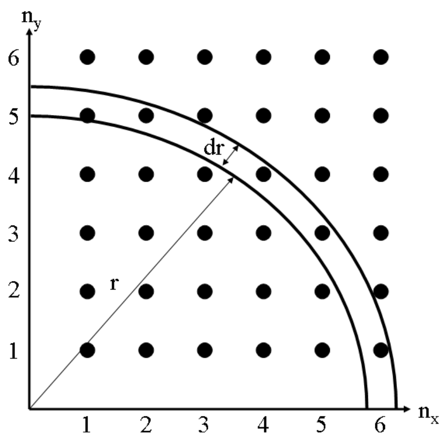

FIGURE 3.10
A view of the x-y plane of a space whose axes are n_x, n_y and n_z where each mode is represented by a dot.

The number of modes having frequencies in the range ν to $\nu + d\nu$ can be calculated by considering the space whose axes are n_x, n_y and n_z. A slice through this space is shown in Figure 3.10. The density of nodes in this space is unity so that the number of nodes per unit radius N_r is the volume defined by a differential spherical octant shell, i.e.

$$N_r(r)dr = \frac{4\pi r^2 dr}{8}. \tag{3.51}$$

The number of nodes per unit frequency is then

$$N_\nu(\nu) = N_r(r)\frac{dr}{d\nu} = 2 \times \frac{4\pi r^2}{8} \times \frac{2l}{c}, \tag{3.52}$$

where the factor of two accounts for the fact that radiation may have two different states of polarization. Combining with Equation 3.50 gives

$$N_\nu(\nu) = \pi \frac{8 l^3 \nu^2}{c^3}. \tag{3.53}$$

In accordance with Boltzmann statistics, the number of oscillators N_n in any given energy state, is a related to the energy of that state E_n through

$$N_n \propto e^{-E_n/k_\mathrm{B}T}. \tag{3.54}$$

The mean energy per oscillator, $\bar{E}$, is given by

$$\bar{E} = \frac{\int_{n=0}^{\infty} E e^{-E/k_\mathrm{B}T}\, dE}{\int_{n=0}^{\infty} e^{-E_n/k_\mathrm{B}T}\, dE} = k_\mathrm{B}T. \tag{3.55}$$

Hence, according to the classical Boltzmann Law each mode has an average energy of $k_\mathrm{B}T$.

The spectral energy density, u_ν in the frequency range ν to $\nu + d\nu$ is then the total energy of all modes in that frequency range divided by the volume of the cavity, i.e.

$$u_\nu(T) d\nu = \frac{N_\nu(\nu)}{l^3} k_\mathrm{B}T = 8\pi k_\mathrm{B}T \frac{\nu^2}{c^3} \tag{3.56}$$

Using the relation between energy density and radiance (Equation 3.43)

$$L_\lambda(\lambda) = \frac{2 k_\mathrm{B} T \nu^2}{c^2} = \frac{8\pi k_\mathrm{B} T}{\lambda^4}. \tag{3.57}$$

This is know as the Rayleigh - Jeans law and it agrees well with experiment for long wavelengths. As the radiance was proportion to λ^{-4} it increases without limit as the wavelength decreased. This was known as the ultraviolet catastrophe.

In 1905 Planck introduced the quantum hypothesis and this solved the difficulty. Planck's first postulate stated that the possible energies of a mode of vibration are quantised as 0, $h\nu$, $2h\nu$, $3h\nu$, ..., where ν is the frequency. Then the mean energy per oscillator, $\bar{E}$, is given by

$$\bar{E} = \frac{\sum_{n=0}^{\infty} E_n e^{-E_n/k_\mathrm{B}T}}{\sum_{n=0}^{\infty} e^{-E_n/k_\mathrm{B}T}} = \frac{\sum_{n=0}^{\infty} nh\nu e^{-nh\nu/k_\mathrm{B}T}}{\sum_{n=0}^{\infty} e^{-nh\nu/k_\mathrm{B}T}} = \frac{h\nu}{e^{-h\nu/k_\mathrm{B}T} - 1}. \tag{3.58}$$

Using this expression instead of $k_\mathrm{B}T$ in Equation 3.57 gives

$$B_\nu(\nu, T) = \frac{2h\nu^3}{c^2 \left(e^{\frac{h\nu}{k_\mathrm{B}T}} - 1 \right)}, \tag{3.59}$$

which is known as Planck's radiation law or the Planck function. This law can also be expressed in terms of wavenumber

$$B_{\tilde{\nu}}(\tilde{\nu}, T) = \frac{2hc^2\tilde{\nu}^3}{\left(e^{\frac{hc\tilde{\nu}}{k_\mathrm{B}T}} - 1 \right)}, \tag{3.60}$$

TABLE 3.3
Planck function and related constants.

Expression	$B_\lambda(\lambda, T) = \frac{c_1}{\lambda^5(e^{c_2/\lambda T}-1)}$	$B_{\tilde{\nu}}(\tilde{\nu}, T) = \frac{c_1 \tilde{\nu}^3}{(e^{c_2 \tilde{\nu}/T}-1)}$
Units	W m^{-2} μm^{-1} sr^{-1}	W m^{-2} (cm^{-1})$^{-1}$ sr^{-1}
c_1	1.191043×10^8 W m^{-2} μm^4	1.191043×10^{-8} W m^{-2} (cm^{-1})$^{-4}$
c_2	1.4387769×10^4 μm K	1.4387769 cm K

or wavelength

$$B_\lambda(\lambda, T) = \frac{2hc^2}{\lambda^5 \left(e^{\frac{hc}{k_B \lambda T}} - 1\right)}, \tag{3.61}$$

The constants used in calculating the Planck function are given in Table 3.3.

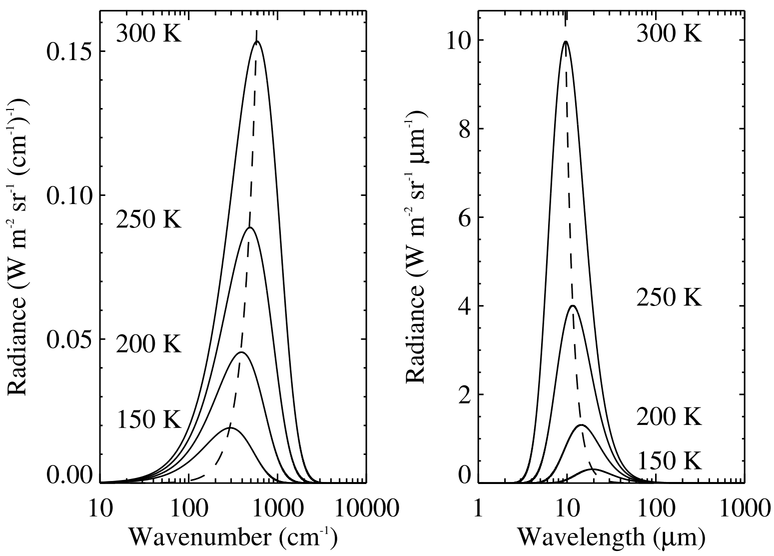

FIGURE 3.11
The solid lines are the Planck curves for black bodies at the temperatures given near the curve peaks. The dashed line is the maximum value of the Planck function determined from Wein's Law.

The Planck function at several temperatures is shown in Figure 3.11 for both $B_{\tilde{\nu}}(\tilde{\nu}, T)$ and $B_\lambda(\lambda, T)$. The energy given off by a black body increases temperature for all wavelengths or wavenumbers. As the temperature increases the spectral value where the Planck function peaks decreases for $B_{\tilde{\nu}}(\tilde{\nu}, T)$ and increases for $B_\lambda(\lambda, T)$. The value of $\lambda_{\max}$ is found by identifying where $\frac{dB_\lambda(\lambda;T)}{d\lambda} = 0$. The derivative is

$$\frac{dB_\lambda(\lambda;T)}{d\lambda} = 2hc^2 \left[\frac{\frac{hc}{k_B\lambda^2 T} e^{\frac{hc}{k_B\lambda T}}}{\lambda^5 \left(e^{\frac{hc}{k_B\lambda T}} - 1\right)^2} - \frac{5}{\lambda^6 (e^{\frac{hc}{k_B\lambda T}} - 1)} \right],$$

giving

$$\frac{hc}{k_B\lambda T \left(1 - e^{-\frac{hc}{k_B\lambda T}}\right)} - 5 = 0.$$

Defining $x \equiv \frac{hc}{k_B\lambda T}$ gives

$$\frac{x}{1 - e^{-x}} - 5 = 0,$$

which can be solved numerically to give $x = 4.965$. Hence

$$\lambda_{\max} = \frac{2898}{T},$$

where $\lambda_{\max}$ is in microns and T is in Kelvin. This expression is known as Wein's Displacement Law. The maxima of $B_{\tilde{\nu}}(\tilde{\nu}, T)$ does not correspond to a maxima in $B_\lambda(\lambda, T)$. A similar derivation gives the wavenumber $\tilde{\nu}_{\max}$ at which the Planck function, $B_{\tilde{\nu}}(\tilde{\nu}, T)$, peaks as

$$\tilde{\nu}_{\max} = 0.5198T.$$

3.3.2 Stefan-Boltzmann Law

Blackbody radiation is isotropic, so the radiant exitance from a blackbody, M^b, is calculated from the Planck function as follows:

$$M^b(+\overline{2\pi}, T) = \int_0^{2\pi} \int_0^\infty \frac{2hc^2}{\lambda^5 \left(e^{\frac{hc}{k_B\lambda T}} - 1\right)} d\lambda\, d\Omega = \pi \int_0^\infty \frac{2hc^2}{\lambda^5 \left(e^{\frac{hc}{k_B\lambda T}} - 1\right)} d\lambda. \quad (3.62)$$

Now making the substitution $x \equiv hc/k_B\lambda T$

$$M^b(+\overline{2\pi}, T) = \frac{2\pi k_B^4 T^4}{h^3 c^2} \int_0^\infty \frac{x^3}{(e^x - 1)} dx. \quad (3.63)$$

The value of the integral is $\pi^4/15$ so

$$M^b(+\overline{2\pi}, T) = \frac{2\pi^5 k_B^4}{15 h^3 c^2} T^4, \quad (3.64)$$

which is usually written as

$$M^{\mathrm{b}}(+\overline{2\pi}, T) = \sigma T^4, \tag{3.65}$$

and known as the Stefan-Boltzmann law where σ is the Stefan-Boltzmann constant:

$$\sigma = \frac{2\pi^5 k_{\mathrm{B}}^4}{15h^3c^2} = 5.670 \times 10^8\ \mathrm{W\,m^{-2}\,K^{-4}}. \tag{3.66}$$

The radiance from a black body is

$$L^{\mathrm{b}}(\boldsymbol{\omega}, T) = \frac{\sigma T^4}{\pi}. \tag{3.67}$$

A black body at the typical temperature of the Earth has a peak emission at $10\,\mu$m, while a blackbody at 6700 K (typical of the Sun's outer atmosphere) has a peak wavelength of $0.55\,\mu$m.

3.3.3 Emission

The emissivity of an object is the ratio of the radiant energy from a body at temperature T to the radiant energy that would be emitted by a perfect blackbody. However this ratio can be expressed in nine different ways based on geometry (directional, conical and hemispherical) and on spectral range (spectral, total and weighted average). Each emissivity term is found from the ratio of the emitted energy integrated over wavelength and/or direction to the equivalent blackbody value. These definitions are shown in Table 3.4. In naming each of the terms the spectral qualifier comes before the directional qualifier so that $\epsilon(\lambda, \boldsymbol{\omega})$ is the spectral directional emissivity, $\epsilon(\lambda, 2\pi)$ is the spectral hemispherical emissivity etc.

When the spectral directional emissivity $\epsilon(\lambda, \boldsymbol{\omega})$ is constant then the emitted radiation is a scaled version of that from a blackbody. An emitter with this property is called a grey body. A coloured body is one where the spectral directional emissivity varies with wavelength.

3.3.4 Brightness temperature

The Planck function can be inverted so that temperature can be expressed as a function of spectral radiance, i.e.

$$T_{\mathrm{B}} = \frac{hc}{k_{\mathrm{B}}\lambda \ln\left(\frac{2hc^2}{\lambda^5 B_\lambda(T)} + 1\right)}, \tag{3.68}$$

or

The temperature so calculated is usually referred to as the brightness temperature. There is no analytical inversion expression for an instrument whose measurement is over a waveband. The brightness temperature in this case is the temperature that gives the measured integrated radiance, L_{measured}, i.e.

$$L_{\mathrm{measured}} = \int_{\lambda_1}^{\lambda_2} \phi(\lambda) B_\lambda(\lambda, T_{\mathrm{B}})\, d\lambda, \tag{3.69}$$

TABLE 3.4
Summary of emissivity terms.

Spectral	Directional	$\epsilon(\lambda,\boldsymbol{\omega}) = \frac{L_\lambda(\lambda,\boldsymbol{\omega})}{B_\lambda(\lambda,T)}$
	Conical	$\epsilon(\lambda,\Delta\omega) = \frac{\int_{\omega_1}^{\omega_2} L_\lambda(\lambda,\boldsymbol{\omega})\,d\Omega}{\int_{\omega_1}^{\omega_2} B_\lambda(\lambda,T)\,d\Omega} = \frac{M_\lambda(\lambda,\Delta\omega)}{M_\lambda^{\mathrm{b}}(\lambda,T,\Delta\omega)}$
	Hemispherical	$\epsilon(\lambda,2\pi) = \frac{\int_0^{2\pi} L_\lambda(\lambda,\boldsymbol{\omega})\,d\Omega}{\int_0^{2\pi} B_\lambda(\lambda,T)\,d\Omega} = \frac{M_\lambda(\lambda,2\pi)}{M_\lambda^{\mathrm{b}}(\lambda,T,2\pi)}$
Band	Directional	$\epsilon(\Delta\lambda,\boldsymbol{\omega}) = \frac{\int_{\lambda_1}^{\lambda_2} L_\lambda(\lambda,\boldsymbol{\omega})\,d\lambda}{\int_{\lambda_1}^{\lambda_2} B_\lambda(\lambda,T)\,d\lambda} = \frac{L(\Delta\lambda,\boldsymbol{\omega})}{L^{\mathrm{b}}(\Delta\lambda,T)}$
	Conical	$\epsilon(\Delta\lambda,\Delta\omega) = \frac{\int_{\lambda_1}^{\lambda_2}\int_{\omega_1}^{\omega_2} L_\lambda(\lambda,\boldsymbol{\omega})\,d\Omega\,d\lambda}{\int_{\lambda_1}^{\lambda_2}\int_{\omega_1}^{\omega_2} B_\lambda(\lambda,T)\,d\Omega\,d\lambda} = \frac{M(\Delta\lambda,\Delta\omega)}{M^{\mathrm{b}}(\Delta\lambda,T,\Delta\omega)}$
	Hemispherical	$\epsilon(\Delta\lambda,2\pi) = \frac{\int_{\lambda_1}^{\lambda_2}\int_0^{2\pi} L_\lambda(\lambda,\boldsymbol{\omega})\,d\Omega\,d\lambda}{\int_{\lambda_1}^{\lambda_2}\int_0^{2\pi} B_\lambda(\lambda,T)\,d\Omega\,d\lambda} = \frac{M(\Delta\lambda,2\pi)}{M^{\mathrm{b}}(\Delta\lambda,T,2\pi)}$
Total	Directional	$\epsilon(\boldsymbol{\omega}) = \frac{\int_0^{\infty} L_\lambda(\lambda,\boldsymbol{\omega})\,d\lambda}{\int_0^{\infty} B_\lambda(\lambda,T)\,d\lambda} = \frac{L(\boldsymbol{\omega})}{L^{\mathrm{b}}}$
	Conical	$\epsilon(\Delta\omega) = \frac{\int_0^{\infty}\int_{\omega_1}^{\omega_2} L_\lambda(\lambda,\boldsymbol{\omega})\,d\Omega\,d\lambda}{\int_0^{\infty}\int_{\omega_1}^{\omega_2} B_\lambda(\lambda,T)\,d\Omega\,d\lambda} = \frac{M(\Delta\omega)}{M^{\mathrm{b}}(T,\Delta\omega)}$
	Hemispherical	$\epsilon(2\pi) = \frac{\int_0^{\infty}\int_0^{2\pi} L_\lambda(\lambda,\boldsymbol{\omega})\,d\Omega\,d\lambda}{\int_0^{\infty}\int_0^{2\pi} B_\lambda(\lambda,T)\,d\Omega\,d\lambda} = \frac{M(2\pi)}{M^{\mathrm{b}}(T,2\pi)}$

where $\phi(\lambda)$ is the instrument response function with limits λ_1 and λ_2 such that

$$\int_{\lambda_1}^{\lambda_2} \phi(\lambda)\,d\lambda = 1.$$

3.4 Transfer of Energy

When radiation is incident upon a layer three processes can occur: absorption, reflection or transmission. The fraction of incident radiant flux Φ^i absorbed, reflected and transmitted is described by the absorptance

$$\mathcal{A} = \frac{\Phi^a}{\Phi^i} \tag{3.70}$$

reflectance

$$\mathcal{R} = \frac{\Phi^r}{\Phi^i} \tag{3.71}$$

and transmittance

$$\mathcal{T} = \frac{\Phi^t}{\Phi^i} \tag{3.72}$$

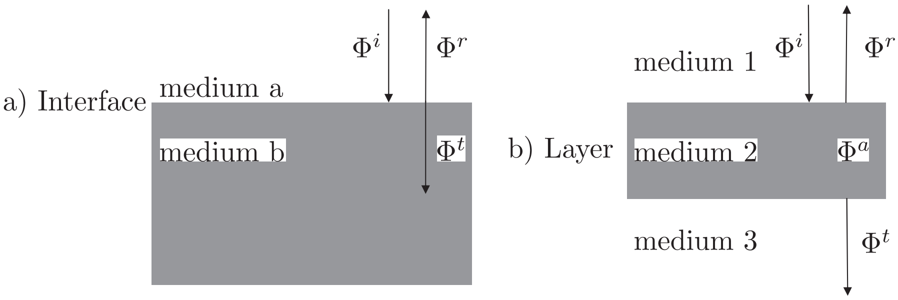

FIGURE 3.12
At an interface (a) radiant flux is either reflected or transmitted where as a layer (b) can reflect, transmit or absorb radiant flux.

where Φ^a is the flux absorbed and Φ^r, Φ^t are the radiant flux due to reflection and transmission respectively. The three terms defined above are dimensionless quantities each of whose values must lie in the interval [0,1]. Reflection and transmission have already been described for an interface between substances of differing refractive index. For a layer the reflectance or transmittance is the describes the distribution of reflected or transmitted light at the layer boundaries which is the cumulative result of redirection of light within the layer by scattering.

The absorbed, reflected and transmitted flux is related to the incident flux by

$$\Phi^i = \Phi^a + \Phi^r + \Phi^t \tag{3.73}$$

so that

$$1 = \mathcal{A} + \mathcal{R} + \mathcal{T} \tag{3.74}$$

represents the conservation of radiant power. Similar expressions can be formed for monochromatic radiation as long as the processes do not alter the wavelength of light (e.g. through fluorescence).

3.5 Absorption

Absorption is the process where radiant energy is removed from the electromagnetic field and converted to some other form of energy e.g. an increase in molecular kinetic energy, a change in atomic excitation or conversion to chemical potential energy through photolysis. Absorption is the complementary process to emission and the two are related by Kirchhoff's Law which can be stated:

> *At thermal equilibrium, the emissivity of a body (or surface) equals its absorptivity.*

TABLE 3.5
Summary of absorbtivity terms.

	Directional	$\alpha(\lambda, \boldsymbol{\omega})$
Spectral	Conical	$\alpha(\lambda, \Delta\omega)$
	Hemispherical	$\alpha(\lambda, 2\pi)$
	Directional	$\alpha(\Delta\lambda, \boldsymbol{\omega})$
Band	Conical	$\alpha(\Delta\lambda, \Delta\omega)$
	Hemispherical	$\alpha(\Delta\lambda, 2\pi)$
	Directional	$\alpha(\boldsymbol{\omega})$
Total	Conical	$\alpha(\Delta\omega)$
	Hemispherical	$\alpha(2\pi)$

To understand this process consider placing an opaque object with spectral hemispherical emissivity $\epsilon(\lambda, 2\pi)$ in an evacuated cavity and allowing the system to reach radiative equilibrium. For this condition to be true the same radiative energy must flow away from the object as impinges on it. The object must be at the same temperature as the walls of the cavity otherwise a system could be constructed that would violate the 2nd law of thermodynamics (i.e. energy would be flowing from a cold object to a hot object). If the walls of the cavity act as a perfect blackbody at temperature T then the amount of irradiance incident on an elemental area of the object would be $\pi B_\lambda(\lambda, T)$. If $\alpha(\lambda, 2\pi)$ of the incident radiance is absorbed then by conservation of energy $\rho(\lambda, 2\pi) = 1 - \alpha(\lambda, 2\pi)$ of the incident irradiance must be reflected away from the object. For radiative balance

$$\pi B_\lambda(\lambda, T) = \epsilon(\lambda, 2\pi)\pi B_\lambda(\lambda, T) + \rho(\lambda, 2\pi)\pi B_\lambda(\lambda, T) \tag{3.75}$$

$$1 = \epsilon(\lambda, 2\pi) + (1 - \alpha(\lambda, 2\pi)) \tag{3.76}$$

$$\alpha(\lambda, 2\pi) = \epsilon(\lambda, 2\pi) \tag{3.77}$$

A similar but slight more complex argument can be invoked to show that the spectral directional absorptivity has the same value as the spectral directional emissivity i.e. $\alpha(\lambda, \boldsymbol{\omega}) = \epsilon(\lambda, \boldsymbol{\omega})$ (see Problem 3.5). Given this fact it follows that Kirchhoff's Law can be used to define the equivalence between the absorptivity terms given in Table 3.5 and the emissivity terms given in Table 3.4.

3.6 Reflection

Reflection deals with the redirection of radiant power at a surface. However as the incident or reflected radiant power can be localised to a particular angle, solid angle or hemisphere there are a large number of reflection terms. Three particular modes of illumination are of particular interest:

- The *diffuse* mode where the light illuminating a surface varies as a function of input direction, $\boldsymbol{\omega}_i$ and is described by the spectral radiance field $L^i_\lambda(\boldsymbol{\omega}_i)$. This is the general case.
- The *isotropic* mode where the light illuminating a surface is perfectly diffuse i.e. $L^i_\lambda(\boldsymbol{\omega}_i)$ is the same for all $\boldsymbol{\omega}_i$. In this case the radiance is related to the downward irradiance through $E(-\overline{2\pi}) = \pi L^i_\lambda(\boldsymbol{\omega}_i)$
- The *unidirectional* mode where the incident light is a spectral irradiance $E^i_\lambda(\boldsymbol{\omega}_i)$ from a single direction $\boldsymbol{\omega}_i$. This is typical of the solar beam illuminating a surface.

In the rest of this book the words *isotropic* or *unidirectional* have been placed alongside formula that have been determined assuming that mode of illumination.

3.6.1 Angular Distribution of Reflection

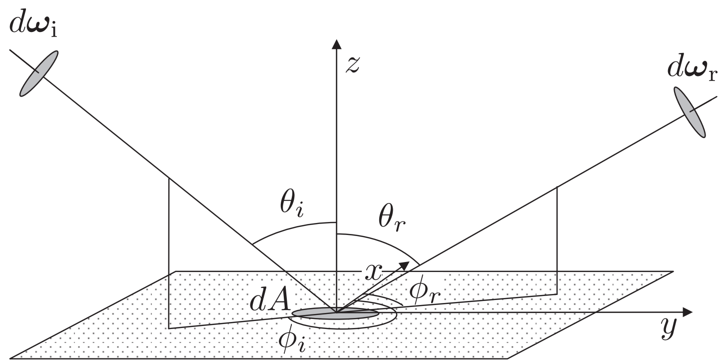

FIGURE 3.13

Reflection geometry.

Consider the reflection of radiation by a surface element dA illuminated by a single small source. The source radiance is L_i and it subtends a solid angle $d\omega_i = \sin\theta_i\, d\theta_i\, d\phi_i$ at dA. The spherical coordinates are orientated so that θ is the angle from the normal to dA and ϕ is the angle in the plane of dA from some arbitrary reference direction. This geometry is shown in Figure 3.13. The radiant power incident on dA from the source is

$$d\Phi^i_\lambda = L^i_\lambda \cos\theta_i\, dA\ \sin\theta_i\, d\theta_i\, d\phi_i = L^i_\lambda \cos\theta_i\, dA\, d\omega_i = L^i_\lambda\, dA\, d\Omega_i \tag{3.78}$$

which is just a rewriting of Equation 3.17 which defines radiance. Any energy that is not transmitted or absorbed is reflected into the hemisphere above dA so that if a small receiver is placed where it subtends a solid angle $d\omega_r = \sin\theta_r \, d\theta_r \, d\phi_r$ at dA then the radiation intercepted by the receiver is

$$d\Phi^r_\lambda = L^r_\lambda \cos\theta_r \, dA \; \sin\theta_r \, d\theta_r \, d\phi_r = L^r_\lambda \cos\theta_r \, dA \, d\omega_r = L^r_\lambda \, dA \, d\Omega_r \tag{3.79}$$

The radiant reflectance $f^*(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r)$ for this geometry is then

$$f^*(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r) = \frac{d\Phi^r_\lambda}{d\Phi^i_\lambda} = \frac{L^r_\lambda dA d\Omega_r}{L^i_\lambda dA d\Omega_i} = \frac{L^r_\lambda d\Omega_r}{L^i_\lambda d\Omega_i} \tag{3.80}$$

which can be rewritten as

$$L^r_\lambda d\Omega_r = f^*(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r) L^i_\lambda d\Omega_i. \tag{3.81}$$

From this expression one can see that $f^*(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r)$ takes the radiant power at dA due to L^i_λ and redirects it into direction $\boldsymbol{\omega}_r$. It is also worth noting the symmetry in this expression so that reflection is a reciprocal process and the same result would be obtained with the source and receiver interchanged.

Equation 3.81 has a limited applicability as it requires knowledge of the differential solid angle of reflection. By defining the bidirectional reflectance distribution function (BRDF) $f^r(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r)$ as the radiant reflectance *per reflected projected solid angle*, i.e.

$$f^r(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r) = \frac{df^*}{d\Omega_r} = \frac{L^r_\lambda}{L^i_\lambda d\Omega_i} = \frac{L^r_\lambda}{E^i_\lambda(-2\pi)}. \qquad [\mathrm{sr}^{-1}] \tag{3.82}$$

the reflected intensity is more simply expressed

$$L^r_\lambda = f^r(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r) L^i_\lambda d\Omega_i. \tag{3.83}$$

As reflection from a surface has the potential to concentrate rays into a small solid angle the BRDF lies in the interval $[0, \infty)$. As both the radiant reflectance and the BRDF are defined in terms of spectral quantities they are themselves spectrally dependent. As emphasised by *Schaepman-Strub et al.* [2006] the BRDF is the ratio of infinitesimal quantities and cannot be directly measured.

For diffuse illumination the reflected radiance is the sum of the contributions from each indent direction $\boldsymbol{\omega}_i$ into the reflected direction $\boldsymbol{\omega}_r$ as

$$L^r_\lambda(\boldsymbol{\omega}_\mathrm{r}) = \int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_\mathrm{i}) f^r(\boldsymbol{\omega}_\mathrm{i}, \boldsymbol{\omega}_\mathrm{r}) \, d\Omega_i. \tag{3.84}$$

Implicit in the equations above is the assumption that reflection is describing incident energy travelling downward (in the sense that z component of the incident radiance direction vector is less than zero) being redirected upward. Equally valid are equations derived for an upward radiation field reflected downward by a layer as the reflectance is the same for a homogeneous layer whether illuminated from above or below.

3.6.2 Reflectance Terms

The BRDF can be used to construct reflectance terms that describe the transfer of energy for a range of geometries. For instance the directional-hemispherical reflectance, $\mathcal{R}(\omega_i, 2\pi)$, is the ratio of the reflected power in the hemisphere above the surface to the power comining from a specific direction. Mathematically it is defined by

$$\mathcal{R}(\omega_i, 2\pi) = \frac{\int_0^{2\pi} \left[L_\lambda^i(\omega_i) f^r(\omega_i, \omega_r) d\Omega_i \right] \, d\Omega_r}{L_\lambda^i(\omega_i) d\Omega_i} = \int_0^{2\pi} f^r(\omega_i, \omega_r) \, d\Omega_r \tag{3.85}$$

The hemispherical-directional reflectance, $\mathcal{R}(-2\pi, \omega_r)$, relates the outgoing radiance to the incoming hemispherical illumination and is defined as

$$\mathcal{R}(-2\pi, \omega_r) = \frac{\int_0^{-2\pi} L_\lambda^i(\omega_i) f^r(\omega_i, \omega_r) \, d\Omega_i}{\int_0^{-2\pi} L_\lambda^i(\omega_i) \, d\Omega_i} \qquad [\mathrm{sr}^{-1}] \tag{3.86}$$

The bihemispherical reflectance, $\mathcal{R}(-2\pi, 2\pi)$, is the ratio of the hemispherically integrated reflected power from the surface to the incident power onto the surface, i.e.

$$\mathcal{R}(-2\pi, 2\pi) = \frac{\int_0^{2\pi} \int_0^{-2\pi} L_\lambda^i(\omega_i) f^r(\omega_i, \omega_r) \, d\Omega_i \, d\Omega_r}{\int_0^{-2\pi} L_\lambda^i(\omega_i) \, d\Omega_i} \tag{3.87}$$

Albedo is often used as a synonym for bihemispherical reflectance. However care has to be taken as albedo is also used as an abbreviation for the Bond albedo which is the fraction of sunlight a planetary body reflects.

3.6.3 Reflection Function

In some instances it is convenient to use a reflection function (or reflectance factor) reflection functionwhich is defined as the reflectance relative to that from an ideal Lambertian surface. The bidirectional reflection function $R(\omega_i, \omega_r)$ is then

$$R(\omega_i, \omega_r) = \frac{f^r(\omega_i, \omega_r)}{1/\pi} \qquad [\text{dimensionless}] \tag{3.88}$$

Using this definition, the reflected radiance is

$$L_\lambda^r(\omega_r) = \frac{1}{\pi} \int_0^{-2\pi} L_\lambda^i(\omega_i) R(\omega_i, \omega_r) \, d\Omega_i. \tag{3.89}$$

It is important to realise that as $R(\omega_i, \omega_r)$ is a ratio it can take on values in the domain $[0, \infty]$. Physically, a reflected direction where $R(\omega_i, \omega_r)$ is larger than one can be interpreted as the surface concentrating the reflected energy into that direction. Table 3.6 relates reflectance terms to the reflection function.

TABLE 3.6
Reflectance Terms defined in terms of the Reflection Function.

bidirectional transmittance distribution function	
	$f^r(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r) = R(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r)/\pi$
hemispherical-directional reflectance	
	$\mathcal{R}(-2\pi, \boldsymbol{\omega}_r) = \frac{1}{\pi}\int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_i) R(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r)\, d\Omega_i / \int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_i)\, d\Omega_i$
directional-hemispherical reflectance	
	$\mathcal{R}(\boldsymbol{\omega}_i, 2\pi) = \frac{1}{\pi}\int_0^{2\pi} R(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t)\, d\Omega_r$
bihemispherical reflectance	
	$\mathcal{R}(-2\pi, 2\pi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_i) R(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t)\, d\Omega_i\, d\Omega_r / \int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_i)\, d\Omega_i$

3.6.4 Special Cases

Generally knowledge of the incident radiation field is needed to evaluate either the hemispherical-directional reflectance or the bihemispherical reflectance. A simplification occurs when the incident radiation field is either unidirectional or isotropically diffuse.

3.6.4.1 Isotropic Incident Radiation

For isotropic radiation L^i_λ is independent of incident direction and the associated irradiance is $E^i_\lambda(-\overline{2\pi}) = \pi L^i_\lambda$. The hemispherical-directional reflectance is then

$$\mathcal{R}(-\overline{2\pi}, \boldsymbol{\omega}_r) = \frac{1}{\pi}\int_0^{-2\pi} f^r(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r)\, d\Omega_i = \frac{1}{\pi^2}\int_0^{-2\pi} R(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r)\, d\Omega_i \quad \text{[isotropic]} \quad (3.90)$$

The hemispherical-directional reflectance for isotropic illumination is used to give the radiance reflected by an isotropically illuminated surface,

$$L^r_\lambda(\boldsymbol{\omega}_r) = E^i_\lambda(-\overline{2\pi})\mathcal{R}(-\overline{2\pi}, \boldsymbol{\omega}_r) \quad \text{[isotropic]} \quad (3.91)$$

For isotropic illumination the bihemispherical reflectance is known as the white sky albedo and is calculated from

$$\mathcal{R}(-\overline{2\pi}, 2\pi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{-2\pi} f^r(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r)\, d\Omega_i\, d\Omega_r = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{-2\pi} R(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r)\, d\Omega_i\, d\Omega_r. \quad \text{[isotropic]} \quad (3.92)$$

This term is useful for calculating the reflected irradiance for isotropic incidence

$$E^r_\lambda(2\pi) = E^i_\lambda(-\overline{2\pi})\mathcal{R}(-\overline{2\pi}, 2\pi). \quad \text{[isotropic]} \quad (3.93)$$

3.6.4.2 Unidirectional Incident Radiation

For unidirectional parallel incident irradiance from direction $\boldsymbol{\omega}_\mathrm{i}$

$$E^i_\lambda(-2\pi) = \cos\theta_\mathrm{i} E^i_\lambda(\boldsymbol{\omega}_\mathrm{i}) \quad \text{[unidirectional]} \quad (3.94)$$

so that the reflected radiance is given by

$$L^r_\lambda(\boldsymbol{\omega}_\mathrm{r}) = \cos\theta_\mathrm{i} E^i_\lambda(\boldsymbol{\omega}_\mathrm{i}) f^r(\boldsymbol{\omega}_\mathrm{i}, \boldsymbol{\omega}_\mathrm{r}) = \cos\theta_i E^i_\lambda(\boldsymbol{\omega}_i)\frac{R(\boldsymbol{\omega}_i, \boldsymbol{\omega}_r)}{\pi}. \quad \text{[unidirectional]} \quad (3.95)$$

For the unidirectional case

$$\mathcal{R}(-2\pi, \boldsymbol{\omega}_\mathrm{r}) = f^r(\boldsymbol{\omega}_\mathrm{i}, \boldsymbol{\omega}_\mathrm{r}) = \frac{R(\boldsymbol{\omega}_\mathrm{i}, \boldsymbol{\omega}_\mathrm{r})}{\pi} \quad \text{[unidirectional]} \quad (3.96)$$

Additionally the directional-hemispherical reflectance has the same value as the bi-hemispherical reflectance i.e.

$$\mathcal{R}(\boldsymbol{\omega}_\mathrm{i}, 2\pi)\,[= \mathcal{R}(-2\pi, 2\pi)] = \int_0^{2\pi} f^r(\boldsymbol{\omega}_\mathrm{i}, \boldsymbol{\omega}_\mathrm{r})\, d\Omega_r = \frac{1}{\pi}\int_0^{2\pi} R(\boldsymbol{\omega}_\mathrm{i}, \boldsymbol{\omega}_\mathrm{r})\, d\Omega_r. \quad \text{[unidirectional]} \quad (3.97)$$

In the first case the radiation field is providing energy from a specific direction, in the second case the reflectance is only being expressed with respect to radiance from that direction. As the directional-hemispherical reflectance represents the energy reflected by a surface for one illumination direction it is sometimes referred to as the black sky albedo. The reflected irradiance for unidirectional incidence is best expressed generally as

$$E^r_\lambda(2\pi) = \cos\theta_i E^i_\lambda(\boldsymbol{\omega}_i)\mathcal{R}(\boldsymbol{\omega}_i, 2\pi). \quad \text{[unidirectional]} \quad (3.98)$$

3.6.4.3 Lambertian Reflector

A Lambertian reflector reflects incident energy isotropically. Its BRDF is therefore $f^r = \rho/\pi$ which is independent of incident or reflection angle and where ρ is a constant in the range $[0, 1]$. An ideal Lambertian reflector redirects all the energy that is incident on it (i.e. $\rho = 1$) so $f^r = 1/\pi$.

The directional-hemispherical reflectance for a Lambertian reflector is

$$\mathcal{R}(\boldsymbol{\omega}_i, 2\pi) = \frac{\int_0^{2\pi} L^i_\lambda(\boldsymbol{\omega}_i)\frac{\rho}{\pi} d\Omega_i\, d\Omega_r}{L^i_\lambda(\boldsymbol{\omega}_i) d\Omega_i} = \rho. \quad \text{[Lambertian]} \quad (3.99)$$

The hemispherical-directional reflectance for a Lambertian reflector is

$$\mathcal{R}(-2\pi, \boldsymbol{\omega}_r) = \frac{\int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_i)\frac{\rho}{\pi}\, d\Omega_i}{\int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_i)\, d\Omega_i} = \frac{\rho}{\pi} \quad \text{[Lambertian]} \quad (3.100)$$

The bihemispherical reflectance for a Lambertian reflector is

$$\mathcal{R}(-2\pi, 2\pi) = \frac{\int_0^{2\pi}\int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_i)\frac{\rho}{\pi}\, d\Omega_i\, d\Omega_r}{\int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_i)\, d\Omega_i} = \rho. \quad \text{[Lambertian]} \quad (3.101)$$

3.7 Transmission

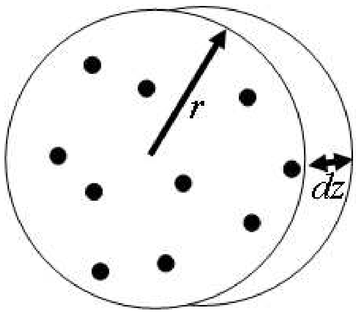

FIGURE 3.14
Attenuation of a beam of light.

Consider a circular beam of light travelling though a volume comprising particles that could remove energy from the beam of cross-section r as shown in Figure 3.14. Each particle has an effective cross-section of σ and there are n particles per unit volume. Then in an infinitesimal distance dz the fraction of the beam that is lost is the ratio between the area the particles present and the area of the beam. The change in radiance in dz is

$$dL = -\frac{\sigma n \pi r^2 \, dz}{\pi r^2} L \tag{3.102}$$

Solving this equation gives Bouguer's Law

$$L(z) = L_0 e^{-\sigma n z} \tag{3.103}$$

The magnitude of the exponential term is known as the optical path χ so

$$\chi = \sigma n z \tag{3.104}$$

The direct transmittance through the media is related to the optical path by

$$\mathcal{T} \left(= \frac{L(z)}{L_0} \right) = e^{-\chi} \tag{3.105}$$

In addition to the direct transmittance energy that is removed from the beam can be scattered one or more times so that it also passes through the media. Whereas the direct transmittance emerges from the media with its direction unaltered the scattered energy can exit the media in any direction. Energy that is transmitted in this way is known as the diffuse transmittance.

3.7.1 Angular Distribution of Diffuse Transmission

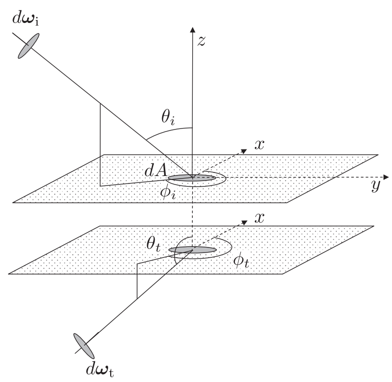

FIGURE 3.15

Transmission geometry.

Consider the transmission of radiation by a layer which is infinitesimally thin physically but has an arbitrary optical thickness. A surface element dA of the layer is illuminated by a single small source radiance L_i than subtends a solid angle $d\omega_i = \sin\theta_i\, d\theta_i\, d\phi_i$ at dA. A small receiver is placed below the layer where it subtends a solid angle $d\omega_r = \sin\theta_r\, d\theta_r\, d\phi_r$ at dA. This geometry is shown in Figure 3.15. The fraction of the radiant energy incident on an elemental area dA that passes is measured by the receiver is the diffuse transmitted energy and it can be represented by expressions analogous to those for diffuse reflectance.

The radiant transmittance is defined as

$$f^{\dagger}(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t) = \frac{d\Phi_\lambda^t}{d\Phi_\lambda^i} = \frac{L_\lambda^t dA d\Omega_t}{L_\lambda^i dA d\Omega_i} = \frac{L_\lambda^t d\Omega_t}{L_\lambda^i d\Omega_i} \tag{3.106}$$

which can be rewritten as

$$L_\lambda^t d\Omega_t = L_\lambda^i f^{\dagger}(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t) d\Omega_i \tag{3.107}$$

The spectral bidirectional transmittance distribution function (BTDF) is then defined by

$$f^t(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t) = \frac{df^{\dagger}}{d\Omega_t} = \frac{L_\lambda^t}{L_\lambda^i d\Omega_i} = \frac{L_\lambda^t}{E_\lambda^i(-2\pi)} \qquad [\mathrm{sr}^{-1}] \tag{3.108}$$

For diffuse illumination the diffusely transmitted radiance is the sum of the contributions from each indent direction $\boldsymbol{\omega}_i$ into the transmitted direction $\boldsymbol{\omega}_t$ as

$$L^t_\lambda(\boldsymbol{\omega}_t) = \int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_i) f^t(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t)\, d\Omega_i. \tag{3.109}$$

The direct component is not included here but contributes to the diffuse transmission to create the total transmission. This is discussed in section 3.7.4.

3.7.2 Transmittance Terms

As with the BRDF the BTDF can be used to construct a number of transmittance terms for the diffusely transmitted light. The principal terms are the hemispherical-directional transmittance, $\mathcal{T}(-2\pi, \boldsymbol{\omega}_r)$, relates the outgoing radiance to the incoming hemispherical illumination and is defined as

$$\mathcal{T}(-2\pi, \boldsymbol{\omega}_t) = \frac{\int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_i) f^t(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t)\, d\Omega_i}{\int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_i)\, d\Omega_i} \tag{3.110}$$

The directional-hemispherical transmittance, $\mathcal{T}(\boldsymbol{\omega}_i, -2\pi)$, is defined by

$$\mathcal{T}(\boldsymbol{\omega}_i, 2\pi) = \frac{\int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_i) f^t(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t)\, d\Omega_t}{L^i_\lambda(\boldsymbol{\omega}_i)} = \int_0^{-2\pi} f^t(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t)\, d\Omega_t. \tag{3.111}$$

and the bihemispherical transmittance, $\mathcal{T}(-2\pi, -2\pi)$, is defined by

$$\mathcal{T}(-2\pi, -2\pi) = \frac{\int_0^{2\pi} \int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_i) f^t(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t)\, d\Omega_i\, d\Omega_t}{\int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_i)\, d\Omega_i} \tag{3.112}$$

Knowledge of the incident radiation field is needed to evaluate the hemispherical-directional transmittance and the bihemispherical transmittance. A simplification occurs when the incident radiation field is either unidrectional or isotropically diffuse.

3.7.2.1 Transmittance Terms for Isotropic Incident Radiation

The hemispherical-directional transmittance $\mathcal{T}(-\overline{2\pi}, \boldsymbol{\omega}_t)$ for isotropic illumination becomes

$$\mathcal{T}(-\overline{2\pi}, \boldsymbol{\omega}_t) = \frac{1}{\pi} \int_0^{-2\pi} f^t(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t)\, d\Omega_i \qquad \text{[isotropic]} \tag{3.113}$$

so that in this case the transmitted radiance is

$$L^t_\lambda(\boldsymbol{\omega}_t) = E^i_\lambda(-\overline{2\pi}) \mathcal{T}(-\overline{2\pi}, \boldsymbol{\omega}_t) \qquad \text{[isotropic]} \tag{3.114}$$

where $E^i_\lambda(-2\pi) = \pi L^i_\lambda$.

For isotropic illumination the bihemispherical transmittance, becomes

$$\mathcal{T}(-\overline{2\pi}, -2\pi) = \frac{1}{\pi} \int_0^{-2\pi} \int_0^{-2\pi} f^t(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t)\, d\Omega_i\, d\Omega_t. \qquad \text{[isotropic]} \quad (3.115)$$

This term is useful for calculating the transmitted irradiance for isotropic incidence

$$\begin{aligned} E^t_\lambda(-2\pi) &= \int_0^{-2\pi} L^t_\lambda(\boldsymbol{\omega}_t)\, d\Omega_t = \int_0^{-2\pi} \int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_i) f^t(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t)\, d\Omega_i\, d\Omega_t \\ &= E^i_\lambda(-\overline{2\pi}) \mathcal{T}(-\overline{2\pi}, -2\pi). \qquad \text{[isotropic]} \quad (3.116) \end{aligned}$$

3.7.2.2 Transmittance Terms for Unidirectional Incident Radiation

For illumination from a single direction

$$L^t_\lambda(\boldsymbol{\omega}_t) = \cos\theta_i E^i_\lambda(\boldsymbol{\omega}_i) f^t(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t) \qquad \text{[unidirectional]} \quad (3.117)$$

The directional-hemispherical transmittance is useful for calculating the transmitted irradiance for unidirectional incidence

$$\begin{aligned} E^t_\lambda(-\overline{2\pi}) &= \int_0^{-2\pi} L^t_\lambda(\boldsymbol{\omega}_t)\, d\Omega_t = \cos\theta_i E^i_\lambda(\boldsymbol{\omega}_i) \int_0^{-2\pi} f^t(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t)\, d\Omega_t, \\ &= \cos\theta_i E^i_\lambda(\boldsymbol{\omega}_i) \mathcal{T}(\boldsymbol{\omega}_i, -\overline{2\pi}). \qquad \text{[unidirectional]} \quad (3.118) \end{aligned}$$

3.7.3 Transmittance Factors

The spectral bidirectional transmittance distribution factor $T(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t)$ describes the transmittance relative to a perfect diffuser ($f^t(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t) = 1/\pi$) and is defined

$$T(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t) = \frac{f^t(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t)}{1/\pi} = \frac{\pi L^t_\lambda}{L^i_\lambda d\Omega_i} \qquad \text{[dimensionless]} \quad (3.119)$$

Using this definition gives the transmitted radiance as

$$L^t_\lambda(\boldsymbol{\omega}_t) = \frac{1}{\pi} \int_0^{-2\pi} L^i_\lambda(\boldsymbol{\omega}_i) T(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t)\, d\Omega_i \qquad (3.120)$$

for diffuse illumination of the surface and

$$L^t_\lambda(\boldsymbol{\omega}_t) = \frac{\cos\theta_i E^i_\lambda(\boldsymbol{\omega}_i) T(\boldsymbol{\omega}_i, \boldsymbol{\omega}_t)}{\pi} \qquad \text{[unidirectional]} \quad (3.121)$$

for a unidirectional beam. Table 3.7 relates transmittance terms to transmittance factors.

TABLE 3.7
Transmittance Terms defined in terms of the Transmittance Factor.

bidirectional transmittance distribution function
$f^t(\omega_i, \omega_t) = T(\omega_i, \omega_t)/\pi$
hemispherical-directional transmittance
$\mathcal{T}(-2\pi, \omega_t) = \frac{1}{\pi} \int_0^{-2\pi} L_\lambda^i(\omega_i) T(\omega_i, \omega_t)\, d\Omega_i / \int_0^{-2\pi} L_\lambda^i(\omega_i)\, d\Omega_i$
directional-hemispherical transmittance
$\mathcal{T}(\omega_i, 2\pi) = \frac{1}{\pi} \int_0^{2\pi} T(\omega_i, \omega_t)\, d\Omega_t$
bihemispherical transmittance
$\mathcal{T}(-2\pi, 2\pi) = \frac{1}{\pi} \int_0^{2\pi} \int_0^{-2\pi} L_\lambda^i(\omega_i) T(\omega_i, \omega_t)\, d\Omega_i\, d\Omega_t / \int_0^{-2\pi} L_\lambda^i(\omega_i)\, d\Omega_i$

3.7.4 Total Transmittance

The added complication for transmittance is that in addition to the energy in the diffuse transmittance there is energy transmitted by the unattenuated beam. The total transmittance, is given by summing these two components. The bidirectional total transmittance distribution function $f^{tt}(\omega_i, \omega_t)$ is defined

$$f^{tt}(\omega_i, \omega_t) = \begin{matrix} f^t(\omega_i, \omega_t) & \omega_i \neq \omega_t \\ f^t(\omega_i, \omega_t) + e^{-\chi/\cos\theta_i} & \omega_i = \omega_t \end{matrix} \quad (3.122)$$

where χ is the optical depth of the transmitting layer. For unidirectional illumination, the irradiance transmitted by such a layer is

$$E_\lambda^t(-2\pi) = \cos\theta_i E_\lambda^i(\omega_i)\mathcal{T}(\omega_i, 2\pi) + \cos\theta_i E_\lambda^i(\omega_i) e^{-\chi/\cos\theta_i}$$
$$\text{[unidirectional]} \quad (3.123)$$

The diffuse fraction f^{d} is the ratio of the diffuse irradiance to the total irradiance illuminating a surface. So for this case

$$f^{\mathrm{d}} = \frac{\cos\theta_i E_\lambda^i(\omega_i)\mathcal{T}(\omega_i, 2\pi)}{\cos\theta_i E_\lambda^i(\omega_i)\mathcal{T}(\omega_i, 2\pi) + \cos\theta_i E_\lambda^i(\omega_i) e^{-\chi/\cos\theta_i}} = \frac{\mathcal{T}(\omega_i, 2\pi)}{\mathcal{T}(\omega_i, 2\pi) + e^{-\chi/\cos\theta_i}}$$
$$\text{[unidirectional]} \quad (3.124)$$

3.8 Scattering

■ This section needs review ■

3.8.1 Scattering in Radiometric Terms

The angular distribution of the scattered radiation is the paramount feature of a scattering volume. The radiance, $L^i_\lambda(\boldsymbol{\omega}_\mathbf{i})$, produces an irradiance, $dE^i_\lambda(\boldsymbol{\omega}_\mathbf{i}) = L^i_\lambda(\boldsymbol{\omega}_\mathbf{i})d\boldsymbol{\omega}_\mathbf{i}$, in the volume. The distribution of radiant intensity, $dI^{\mathrm{s}}_\lambda(\boldsymbol{\omega}_\mathbf{s})$, scattered from dV is specified by the volume scattering function defined by

$$f^{\mathrm{sca}}(\lambda; \boldsymbol{\omega}_\mathbf{i}; \boldsymbol{\omega}_\mathbf{s}) = \frac{d^2 I^{\mathrm{s}}_\lambda(\boldsymbol{\omega}_\mathbf{s})}{dE^{\mathrm{i}}_\lambda(\boldsymbol{\omega}_\mathbf{i})dV}, \tag{3.125}$$

$$= \frac{d^2 I^{\mathrm{s}}_\lambda(\boldsymbol{\omega}_\mathbf{s})}{dL^{\mathrm{i}}_\lambda(\boldsymbol{\omega}_\mathbf{i})d\boldsymbol{\omega}_\mathbf{i}dV}. \tag{3.126}$$

Although the volume scattering function provides the angular distribution of scattered radiation, it is not very useful because its magnitude is dependent on the incident irradiance and the volume size.

To remove the dimensionality the *phase function* is defined as the ratio of the scattered intensity to the intensity from an isotropic scatterer.

$$P(\lambda; \boldsymbol{\omega}_\mathbf{i}; \boldsymbol{\omega}_\mathbf{s}) = 4\pi \frac{I^{\mathrm{s}}_\lambda(\boldsymbol{\omega}_\mathbf{s})}{\int_0^{4\pi} I^{\mathrm{s}}_\lambda(\boldsymbol{\omega}_\mathbf{s})\, d\boldsymbol{\omega}_\mathbf{s}.}, \tag{3.127}$$

The later can be found from equating the scattered energy defined by Equation 3.142 to the integral of the scattered intensity so that

$$P(\lambda; \boldsymbol{\omega}_\mathbf{i}; \boldsymbol{\omega}_\mathbf{s}) = 4\pi \frac{I^{\mathrm{s}}_\lambda(\boldsymbol{\omega}_\mathbf{s})}{\int_0^{4\pi} I^{\mathrm{s}}_\lambda(\boldsymbol{\omega}_\mathbf{s})\, d\boldsymbol{\omega}_\mathbf{s}.}, \tag{3.128}$$

For scatterers that have rotational symmetry such as cloud water droplets the phase function can be characterised in terms of the scattering angle

In terms of spherical polar coordinates the incident $\vec{x}_i$ and scattered $\vec{x}_s$ directions are expressed

$$\vec{x}_i = \sin\theta_i \cos\phi_i \vec{\imath} + \sin\theta_i \sin\phi_i \vec{\jmath} + \cos\theta_i \vec{k} \tag{3.129}$$

$$\vec{x} = \sin\theta \cos\phi \vec{\imath} + \sin\theta \sin\phi \vec{\jmath} + \cos\theta \vec{k} \tag{3.130}$$

where $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$ are the unit vectors in the x,y and z directions respectively. Figure 3.16 shows the relevant geometry. The cosine of the scattering angle Θ is found from the dot product of the two vectors i.e.

$$\cos\Theta = \sin\theta_i \cos\phi_i \sin\theta \cos\phi + \sin\theta_i \sin\phi_i \sin\theta \sin\phi + \cos\theta_i \cos\theta \tag{3.131}$$

$$= \cos\theta_i \cos\theta + \sin\theta_i \sin\theta \cos(\phi_i - \phi) \tag{3.132}$$

$$= \mu_i \mu + \sqrt{1-\mu_i^2}\sqrt{1-\mu^2}\cos(\phi_i - \phi) \tag{3.133}$$

Note that the incident vector points to where the light is direction that the light is travelling. Typically we are given the coordinates pointing towards the Sun i.e. θ_o

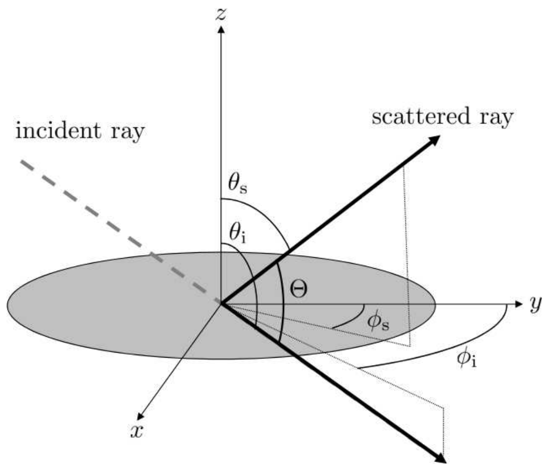

FIGURE 3.16
Relation between angles in the scattering place.

and ϕ_o are the solar zenith and azimuth angles. In this case the scattering angle is

$$\cos\Theta = -\sin\theta_o\cos\phi_o\sin\theta\cos\phi - \sin\theta_o\sin\phi_o\sin\theta\sin\phi - \cos\theta_o\cos\theta \quad (3.134)$$
$$= -\cos\theta_o\cos\theta - \sin\theta_o\sin\theta\cos(\phi_o - \phi) \quad (3.135)$$
$$= -\mu_o\mu - \sqrt{1-\mu_o^2}\sqrt{1-\mu^2}\cos(\phi_o - \phi) \quad (3.136)$$

The scattering angle, Θ is related to the incoming and scattered direction through

$$\cos\Theta = \cos\theta_i\cos\theta_s + \sin\theta_i\sin\theta_s(\phi_s - \phi_i) \quad (3.137)$$

Other notation

For a volume the redirection of irradiance $E_\lambda(\omega_i)$ from direction ω_i to direction ω_s is expressed

$$L_\lambda(\omega_s) = \beta^{\text{sca}}\frac{P(\omega_i,\omega_s)}{4\pi}E_\lambda(\omega_i) \quad (3.138)$$

which can also be expressed in terms of the incident radiance $L_\lambda(\omega_i)$ as

$$L_\lambda(\omega_s) = \beta^{\text{sca}}\frac{P(\omega_i,\omega_s)}{4\pi}L_\lambda(\omega_i)\,d\Omega_i \quad (3.139)$$

The later can be found from equating the scattered energy defined by Equation 3.142 to the integral of the scattered intensity

$$\beta^{\text{sca}}E_\lambda^i = \int_0^{4\pi} I_\lambda^{\text{s}}(\boldsymbol{\omega}_\mathbf{s})\,d\boldsymbol{\omega}_\mathbf{s}. \quad (3.140)$$

The equivalent isotropic intensity is $\beta^{\text{sca}}E_\lambda^i/4\pi$ and the phase function can be exprress as

$$P(\lambda;\boldsymbol{\omega}_\mathbf{i};\boldsymbol{\omega}_\mathbf{s}) = 4\pi\frac{I_\lambda^{\text{s}}(\boldsymbol{\omega}_\mathbf{s})}{\beta^{\text{sca}}E_\lambda^i}, \quad (3.141)$$

3.9 Scattering Cross Section

The scattering cross-section, σ^{sca}, is the equivalent area of the incident beam that intercepts the same energy as that scattered by the particle, i.e.

$$\sigma^{\mathrm{sca}} = \frac{\int_0^{4\pi} I_\lambda^{\mathrm{s}}(\boldsymbol{\omega}_{\mathbf{s}})\, d\boldsymbol{\omega}_{\mathbf{s}}}{E_\lambda^i}, \tag{3.142}$$

where $I_\lambda^{\mathrm{s}}(\boldsymbol{\omega}_{\mathbf{s}})$ represents the intensity of light scattered into the elemental solid angle $\boldsymbol{\omega}_{\mathbf{s}}$ from a beam of light with irradiance E_λ^i.

3.10 Volume Emission Function, Volume Scattering Function

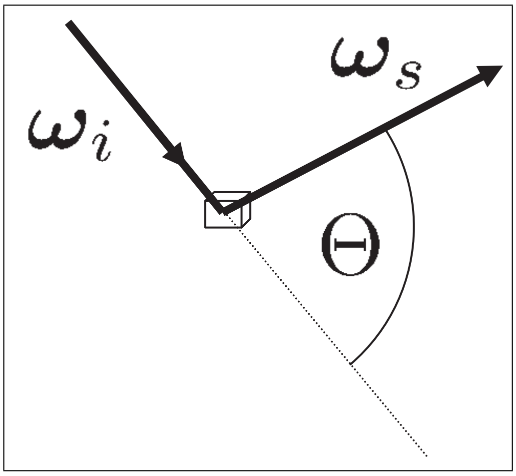

FIGURE 3.17
An illuminated volume.

The volume emission coefficient in scattering, j^{sca}, is the radiant intensity per unit volume in the direction $\boldsymbol{\omega}_{\mathbf{s}}$ which results from radiation scattered from all directions by the volume dV. Hence

$$j^{\mathrm{sca}}(\boldsymbol{\omega}_{\mathbf{s}}) = \int_0^{4\pi} L_\lambda^i(\lambda; \boldsymbol{\omega}_{\mathbf{i}}) f^{\mathrm{sca}}(\lambda; \boldsymbol{\omega}_{\mathbf{i}}; \boldsymbol{\omega}_{\mathbf{s}})\, d\boldsymbol{\omega}_{\mathbf{i}}. \tag{3.143}$$

This may be interpreted as the radiance per unit path length into direction $\boldsymbol{\omega_s}$. The actual radiance is given by the scattering source function, $L^s_\lambda(\lambda;\boldsymbol{\omega_s})$, defined as

$$L^s_\lambda(\lambda;\boldsymbol{\omega_s}) = \frac{j^{sca}}{\beta^{sca}}, \tag{3.144}$$

$$= \int_0^{4\pi} L^i_\lambda(\lambda;\boldsymbol{\omega_i}) \frac{f^{sca}(\lambda;\boldsymbol{\omega_i};\boldsymbol{\omega_s})}{\beta^{sca}}\, d\boldsymbol{\omega_i}, \tag{3.145}$$

which by substitution of Equation 3.162 gives

$$L^s_\lambda(\lambda;\boldsymbol{\omega_s}) = \frac{1}{4\pi}\int_0^{4\pi} L^i_\lambda(\lambda;\boldsymbol{\omega_i}) P(\lambda;\boldsymbol{\omega_i};\boldsymbol{\omega_s})\, d\boldsymbol{\omega_i}. \tag{3.146}$$

It is assumed that the scattering characteristics are independent of the orientation of the volume: for example β^{sca} should strictly be written $\beta^{sca}(\boldsymbol{\omega_i})$. The assumption that the scattering medium is isotropic in respect of the incident direction is generally true for atmospheric scattering — with the exception of an air parcel containing very large drops or ice crystals, both of which make the volume angularly dependent.

3.10.1 Scattering by a Particle

Scattering is the process by which a particle or scattering volume in the path of an electromagnetic wave continuously removes energy from the incident wave and re-radiates the energy into the sphere centred at the particle.

$$I(\Theta) = E\sigma^{sca}\frac{P(\Theta)}{4\pi} \tag{3.147}$$

If all the energy intercepted by a particle, of radius r, was scattered then one might expect the scattering cross-section to be simply πr^2. However this is not the case — as described later, light passing the particle at a distance slightly greater than r can be influenced by the particle. Hence it is useful to introduce the scattering efficiency Q^{sca}, defined by

$$Q^{sca} = \frac{\sigma^{sca}}{\pi r^2}. \tag{3.148}$$

Similarly, the absorption cross section, σ^{abs}, represents the equivalent area of the beam that a particle removes by absorption and Q^{abs} denotes the efficiency of this process where

$$Q^{abs} = \frac{\sigma^{abs}}{\pi r^2}. \tag{3.149}$$

The processes of scattering and absorption are additive and produce an extinction cross-section according to

$$\sigma^{ext} = \sigma^{abs} + \sigma^{sca}, \tag{3.150}$$

and a corresponding extinction efficiency,

$$Q^{ext} = \frac{\sigma^{ext}}{\pi r^2}. \tag{3.151}$$

The ratio of the flux scattered to that scattered and absorbed is represented by the single scatter albedo

$$\tilde{\omega} = \frac{\sigma^{sca}}{\sigma^{ext}}. \tag{3.152}$$

3.10.2 Scattering by a Volume Containing a Distribution of Particles

■ This section needs reordering ■

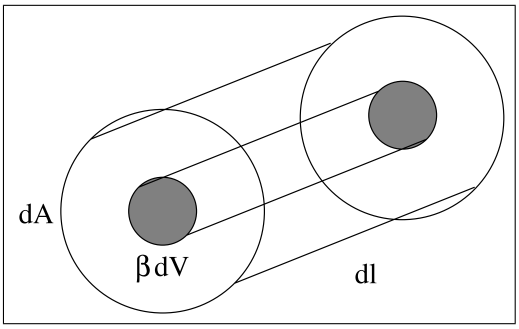

FIGURE 3.18
An illuminated volume.

Consider a volume element, $dV = dAdl$, of a scattering and absorbing medium in thermal equilibrium, irradiated by radiation specified by zenith angle, θ, and azimuth angle ϕ (see Figure 3.18). The terms volume absorption coefficient, β^{abs}, and volume scattering coefficient, β^{sca}, can be thought of as a cross sectional area per unit volume with which the ray interacts by absorption or scattering respectively. The volume extinction coefficient, β^{ext}, gives the cross-sectional area removed from the beam per unit volume. The volume coefficients are related to the cross sections by

$$\beta^{abs} = N\sigma^{abs}, \tag{3.153}$$
$$\beta^{sca} = N\sigma^{sca}, \tag{3.154}$$
$$\beta^{ext} = N\sigma^{ext}, \tag{3.155}$$

where N denotes the number of scatterers/absorbers per unit volume. It follows from Equation 3.150 that

$$\beta^{ext} = \beta^{abs} + \beta^{sca}. \tag{3.156}$$

It is clear that β^{ext} has dimensions of [L^{-1}], usually km^{-1}. It is also useful to use the mass extinction coefficient (sometimes called the mass extinction cross-section), k^{ext} which is related to the volume extinction coefficient by

$$\beta^{\text{ext}} = \rho k^{\text{ext}}, \tag{3.157}$$

where ρ denotes the mass per unit volume, i.e. the density. Similarly

$$\beta^{\text{abs}} = \rho k^{\text{abs}}, \tag{3.158}$$

$$\beta^{\text{sca}} = \rho k^{\text{sca}}. \tag{3.159}$$

Warning: many authors leave out 'mass' and 'volume' and just use, for example, absorption coefficient. If in doubt do a dimensional analysis.

If the volume scatter albedo, $\tilde{\omega}$ is defined to be

$$\tilde{\omega} = \frac{\beta^{\text{sca}}}{\beta^{\text{ext}}}, \tag{3.160}$$

so

$$1 - \tilde{\omega} = \frac{\beta^{\text{abs}}}{\beta^{\text{ext}}}, \tag{3.161}$$

i.e.

$$P(\lambda; \boldsymbol{\omega}_{\mathbf{i}}; \boldsymbol{\omega}_{\mathbf{s}}) = \frac{f^{\text{sca}}(\lambda; \boldsymbol{\omega}_{\mathbf{i}}; \boldsymbol{\omega}_{\mathbf{s}})}{\beta^{\text{sca}}/4\pi}, \tag{3.162}$$

we equate the power in the integral of the scattered radiation with the fraction of the beam that has been scattered

Using Equations 3.142 and 3.125 we can express the volume scattering coefficient in terms of the volume scattering function as

For a unidirectional source Equation 3.125 can be rearranged to give

$$f^{\text{sca}}(\lambda; \boldsymbol{\omega}_{\mathbf{i}}; \boldsymbol{\omega}_{\mathbf{s}}) = \frac{d^2 I^{\text{s}}_{\lambda}(\boldsymbol{\omega}_{\mathbf{s}})}{dL^{\text{i}}_{\lambda}(\boldsymbol{\omega}_{\mathbf{i}}) d\boldsymbol{\omega}_{\mathbf{i}} dV}. \tag{3.163}$$

which shows that β^{sca} can be interpreted as the fraction of the irradiance scattered per unit distance. If this radiance is scattered isotropically then the scattered radiance is

$$\frac{dI^{\text{s}}_{\lambda}(\boldsymbol{\omega}_{\mathbf{s}})}{dL^{\text{i}}_{\lambda}(\boldsymbol{\omega}_{\mathbf{i}})} = \beta^{\text{sca}}/4\pi E^{\text{i}}_{\lambda} \tag{3.164}$$

The Mie solution for a single sphere can be extended to give the scattering characteristics of a volume containing many spheres. The three important features of an elemental scattering volume are: the volume extinction coefficient, the single scatter albedo, and the phase function. The first of these parameters determines the depth of penetration of unscattered radiation into the medium; the second determines the relative importance of scattering to absorption; while the third gives the directional characteristic of the scattered light.

It is straightforward to extend the Mie solution for one particle to a polydispersion. We assume that the particles are sufficiently far from each other that the distance between them is much greater then the incident wavelength.

For a collection of particles, the volume coefficient or the total cross section area per unit volume is the sum of the cross section times the density of each of the species present. This is written explicitly for a continuous distribution of particle sizes as

$$\beta^x = \int_{r_1}^{r_2} \sigma^x n(r)\, dr, \tag{3.165}$$

where r_1 and r_2 represent the limits of the particle size distribution and $n(r)$ denotes the number density of particles having a radius between r and $r+dr$. The superscript, x, denotes either absorption, scattering, or extinction.

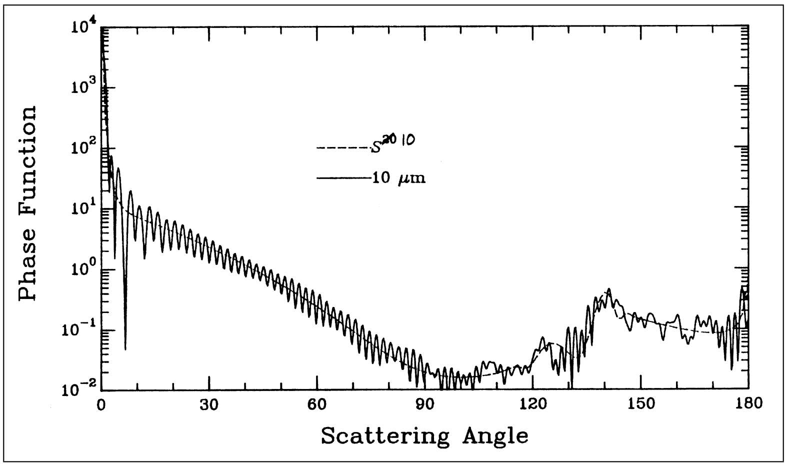

FIGURE 3.19
The phase function for a single drop and for a distribution of drops with an equivalent effective radius.

The volume phase function is expressed as

$$p(\Theta) = \frac{1}{k^2} \frac{\int_{r_1}^{r_2} [(i_1(x;m;\Theta) + i_2(x;m;\Theta))/2]\, n(r)\, dr}{\beta^{\text{sca}}/4\pi}. \tag{3.166}$$

It is evident from this equation that $p(\Theta)$ is independent of the particle concentration, that it is dimensionless, and that it meets the normalisation requirement. The integrated phase function varies smoothly compared to the rapid oscillations which occur in the phase function for a single particle (see Figure 3.19). The smooth change exhibited for a polydisperse drop distribution can be attributed to the contributions from many particles cancelling the effects that are not invariant in angle — hence features such as the forward diffraction peak and the rainbow are retained.

3.10.3 Asymmetry Parameter and Backscatter Fraction

The asymmetry parameter is often used as a single measure of the directional properties of scattered light. The asymmetry parameter has a value from -1, for strictly backscattering, to 1 for strictly forward scattering. For isotropic scattering g is 0.

3.10.4 Legendre Expansion of the Phase Function

In many radiative transfer applications it is convenient to express the phase function as a finite expansion in Legendre polynomials

$$p(\mu) = \sum_{l=0}^{L} \omega_l P_l(\mu), \tag{3.167}$$

where μ the cosine of the scattering angle and $P_l(\mu)$ is a Legendre polynomial of order l. An alternative version of the expansion in Equation 3.167 is [*Wiscombe*, 1977]

$$p(\mu) = \sum_{l=0}^{L} (2l + 1)\chi_l P_l(\mu), \tag{3.168}$$

where χ_l are the normalised Legendre coefficients which are related to ω_l by

$$\chi_l = \frac{\omega_l}{2l + 1}. \tag{3.169}$$

The asymmetry parameter g is the first moment and is related to the Legendre coefficient, ω_1, by $g = \chi_1 = \omega_l/3$.

Fowler [1983], *Allen* [1974], and *Chu and Churchill* [1955] showed how the Legendre coefficients ω_l could be expressed directly from the Mie coefficients a_n and b_n. However their algorithm for the calculation of the coefficients involves the summation of a series within a series and takes considerable time if $x > 15$. For large x it is faster to calculate the Legendre coefficients from the phase function, i.e.

$$\omega_l = \frac{2l + 1}{2} \int_{-1}^{1} p(\mu) P_l(\mu)\, d\mu \qquad \left(\text{or} \quad \chi_l = \frac{1}{2} \int_{-1}^{1} p(\mu) P_l(\mu)\, d\mu\right) \tag{3.170}$$

The computation time of the Legendre coefficients using Equation 3.170 is a function of the number of angles at which the Mie phase function is evaluated. Only one set of points is chosen to calculate all the Legendre coefficients. Earlier authors used Gauss-Legendre quadrature on the interval [-l,l]. If the phase function is a polynomial of degree L the integrand in Equation 3.170 is at most a polynomial of degree $2L$. As Gauss-Legendre quadrature of order N is exact for polynomials of degree less than $2N$, the order of the quadrature formula must exceed the number of significant Legendre coefficients. To improve the accuracy of calculations *Hunt* [1970] recommend using Lobatto quadrature on [-l,l] so that the 0°and l80°scattering angles are included as explicit quadrature points [also see *Kattawar et al.*, 1973]. *Wiscombe*

[1977] noted that a further dramatic improvement in accuracy is gained by using Lobatto quadrature on the interval [0,180] as more quadrature points are included in the forward scattering peak.

In general Eq. [a.a.5] gives the first N moments to about the same accuracy, provided N exceeds some lower bound dependent on the phase function. This means that ω_0 an excellent error monitor. If it equals unity to k decimal places then the remaining ω_l will be accurate to between $k-1$ and $k+l$ decimal places.

Clark et al. [1957] showed that the number of significant terms in the expansion of the phase function is dependent on the drop's size parameter. The expansion of the complex amplitudes in terms of functions related to the Legendre functions requires a little over x terms. This is a consequence of the localization principle which attributes the terms with subscript n to a ray passing at a distance $n\lambda/2\pi$ from the centre. In squaring to obtain the phase function the required number of terms goes to $2x$+a few. However, a polymorphic collection of drops has a much smoother phase function with a strong forward peak. *van de Hulst* [1980] suggested that 20 to 30 Legendre terms would describe the smoothed pattern for water drops independently of their size. Substituting this into the expansion of the phase function and using the addition theorem for spherical harmonics (ref Liou or math book) gives

3.10.5 Approximation of the Forward Peak of the Phase Function

However, a polymorphic collection of drops has a much smoother phase function with a strong forward peak. *van de Hulst* [1980] suggested that 20 to 30 Legendre terms would describe the smoothed pattern for water drops independently of their size. In this approach the forward peak is truncated and the optical depth scaled [*McKellar and Box*, 1981]. The Legendre expansion of the phase function (Equation ref) is

$$P'(\cos\Theta) = 2f\delta_0,(1-\cos\Theta) + (1-f)\sum(2n+1)\chi'_n P_n(\cos\Theta) \tag{3.171}$$

If χ'_n s defined

$$\chi'_n = \frac{\chi_n - f}{1-f} \tag{3.172}$$

then P and P' are identical.

When the new phase function is substituted into the equation of radiative transfer it is found that the optical depth and single scatter albedo are transformed to

$$\tau' = (1-\tilde{\omega}f)\tau \tag{3.173}$$

$$\tilde{\omega}' = \frac{\tilde{\omega}(1-f)}{1-\tilde{\omega}f} \tag{3.174}$$

3.10.6 The Henyey-Greenstein Phase Function

Henyey and Greenstein [1941] introduced an analytic phase function given by

$$p(\cos\Theta) = \frac{1-g^2}{(1+g^2-2g\cos\Theta)^{3/2}}. \tag{3.175}$$

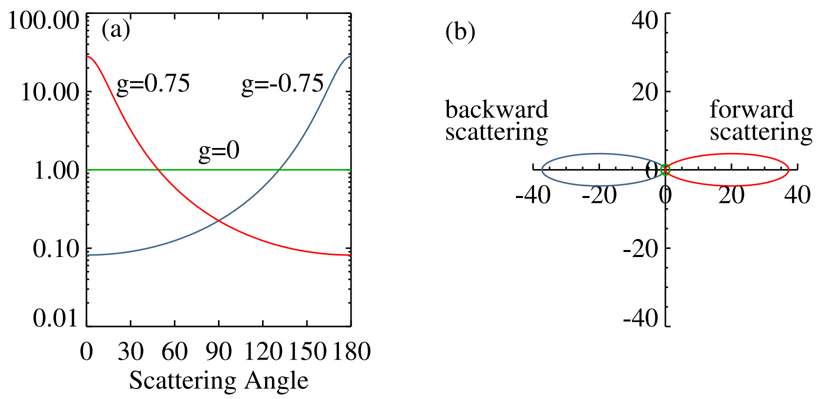

FIGURE 3.20
(a) Henyey-Greenstein phase functions for $g = -0.75$, 0 and 0.75. (b) Polar plot of the same phase functions for an incident ray travelling in the positive 'x' direction.

which can be respressed when $\theta = 0$ or π as

$$p(1) = \frac{1+g}{(1-g)^2} \qquad p(-1) = \frac{1-g}{(1+g)^2}. \tag{3.176}$$

The shape of the phase function, shown in Figure 3.20, is controlled by the parameter g. When g is 1 the Henyey-Greenstein phase function evaluates to zero at all angles except the forwardscatter angle $\Theta = 0$. Similarly when $g = -1$ the only non-zero value is at the backscatter angle $\Theta = \pi$. However these two delta functions are atypical as in general, the shape of the function is similar to phase functions elaborately calculated from Mie theory. However, features such as the corona, rainbow and glory are smoothed out. The influence of anisotropy can be tested in multiple scattering calculations through the variation of the asymmetry parameter, g. Because of this, the Henyey-Greenstein phase function has become a standard for test calculations on multiple scattering.

One further advantage is that the Legendre expansion of the function has a very simple form, i.e.

$$\omega_l = (21 + 1)g^l. \tag{3.177}$$

The Legendre terms for the Henyey-Greenstein phase function are monotonically decreasing for $0 < g < 1/3$. For higher values of g (< 1) the expansion terms first increase then decrease for $l > -1/\ln g - 1/2$ so that $\omega_l \to 0$ as $l \to \infty$. Similarly, the Legendre expansion terms of a phase function calculated using Mie theory (generally) increase then decrease [*Wiscombe*, 1977]. For the same level of accuracy, the expansion limit of a Mie phase function is usually greater than the limit for a Henyey-Greenstein phase function with the same asymmetry parameter. This is a consequence of the smoother shape of the analytic function.

Problem 3.1 Show that the étendue is conserved for an optical system comprising an aperture-lens-aperture.

Problem 3.2 Consider an isotropic radiance field in an medium of refractive index n encountering an interface where the second medium is a characterised by a refractive index of 1 (e.g. this could by light from the ocean encountering the surface). Show that the irradiance is reduced by a factor of n^2.

Problem 3.3 A 60 W isotropic light source is used to illuminate an A4 sheet (21 × 29.7 cm) of paper lying on a desk 3 m away. If a line from the light to the centre of the paper makes an angle of 45°with the paper, what is the radiant flux falling on the sheet.

Problem 3.4 Show that the approximations for the Planck function at high and low frequency extremes are

$$B_\nu(\nu, T) \approx \begin{matrix} \frac{2k_\mathrm{B}T\nu^2}{c^2} & h\nu \ll k_\mathrm{B}T \\ \frac{2h\nu^3}{c^2}e^{-\frac{h\nu}{k_\mathrm{B}T}} & h\nu \gg k_\mathrm{B}T \end{matrix}$$

These two limits are known as the Rayleigh-Jeans and Wein approximations respectively.

Problem 3.5 Show that $\alpha(\lambda, \boldsymbol{\omega}) = \epsilon(\lambda, \boldsymbol{\omega})$.
Hint: One approach is to show that the principle of reciprocity is broken if $\alpha(\lambda, \boldsymbol{\omega}) \neq \epsilon(\lambda, \boldsymbol{\omega})$.

where $\delta_{\boldsymbol{\omega}_i,\boldsymbol{\omega}_t}$ is the Kronecker delta defined as

$$\delta_{\boldsymbol{\omega}_i,\boldsymbol{\omega}_t} \begin{cases} 1 & \boldsymbol{\omega}_i = \boldsymbol{\omega}_t \\ 0 & \boldsymbol{\omega}_i \neq \boldsymbol{\omega}_t \end{cases} \tag{3.178}$$

Problem 3.6 Show that the Henyey-Greenstein phase function is normalised to 1.

Additional Reading

McCluney, R., *Introduction to Radiometry and Photometry*, Artech House, Boston, 1994.
Wolfe, W. L., *Introduction to Radiometry*, SPIE Optical Engineering Press, 1998.