5

Electromagnetic Scattering

Scattering is the process by which a particle in the path of an electromagnetic wave continuously removes energy from the incident wave and re-radiates the energy into the total solid angle centred at the particle. We only consider the far field solution for a single homogeneous sphere. The full formal solution called *Mie theory* is required for a particle that has a similar radius relative to the wavelength of the incident light. This is generally the case for cloud and aerosol particles at visible and near infrared wavelengths. This theory is the basis for describing the rainbow, glory and other atmospheric optical phenomena. When the wavelength of light is large compared to the particle size, Mie theory is well approximated by *Rayleigh scattering*. In the atmosphere molecules act as Rayleigh scatters — this phenomenon explains why the sky is blue.

Scattering properties vary rapidly with wavelength so that broadband approximations are inapplicable. Scattering is always considered as a monochromatic process, and calculations over a bandwidth involve wavelength integration. Variation in scattering properties for particle distributions is often slow enough that a single calculation can suffice for a range of wavelengths.

5.1 Rayleigh Scattering

5.1.1 The Hertzian Dipole

A Hertzian dipole consists of two small spherical conductors carrying equal and opposite charge q separated by a distance l. A thin wire connects the two spheres as shown in Figure 5.1. An alternating current is set up in the wire such that the charge oscillates at an angular frequency ω i.e.

$$q(t) = q_0 \cos \omega t. \tag{5.1}$$

A Cartesian or polar coordinate system can be adopted which has the centre of the dipole at the origin and the spheres located on the z axis. The dipole moment, $\mathbf{p}$ is defined by

$$\mathbf{p} = q\mathbf{l} = q_0 \mathbf{l} \cos \omega t. \tag{5.2}$$

An accelerating charge gives rise to electromagnetic radiation. Hence an oscillating charge produces an electromagnetic field. If the wavelength λ of any periodic

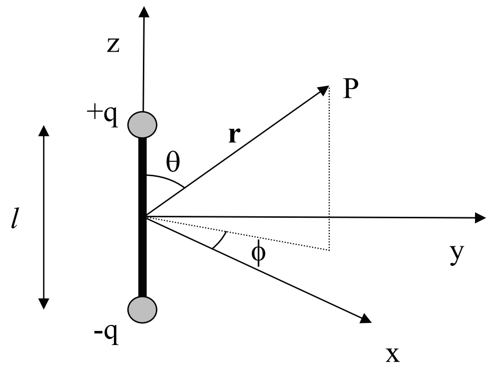

FIGURE 5.1
Dipole.

time variations is such that $l \ll \lambda$ then the radiation from an oscillating dipole at large distances is described by

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\omega^2 p_0 \sin\theta}{4\pi\epsilon_0 c^2 r} e^{i(\omega r/c - \omega t)} \hat{\boldsymbol{\theta}} \tag{5.3}$$

$$\mathbf{B}(\mathbf{r}, t) = -\frac{\omega^2 p_0 \sin\theta}{4\pi\epsilon_0 c^3 r} e^{i(\omega r/c - \omega t)} \hat{\boldsymbol{\phi}} \tag{5.4}$$

These are expressions are derived in Appendix **??** and describe a spherical wave radiating out from the origin whose amplitude falls off as $1/r$.

The instantaneous power per unit area radiated by the dipole is found by evaluating the Poynting vector, **S**, defined in Equation 2.95

$$\mathbf{S}(\mathbf{r}, t) = \frac{\mathcal{R}[\mathbf{E}(\mathbf{r}, t)] \times \mathcal{R}[\mathbf{B}(\mathbf{r}, t)]}{\mu_0} = \frac{\omega^4 p_0^2 \sin^2\theta}{16\pi^2 r^2 \epsilon_0 c^3} \cos^2(\omega r/c - \omega t)\, \hat{\mathbf{r}}. \tag{5.5}$$

The average power per unit area is found by integrating the instantaneous power over one cycle i.e

$$\bar{\mathbf{S}}(\mathbf{r}, t) = \frac{\omega^4 p_0^2 \sin^2\theta}{32\pi^2 r^2 \epsilon_0 c^3}\, \hat{\mathbf{r}}. \tag{5.6}$$

Integrating this over the sphere centred at the dipole gives the total energy radiated per unit time as

$$P = \int_0^{2\pi} \int_0^{\pi} \frac{\omega^4 p_0^2 \sin^2\theta}{32\pi^2 r^2 \epsilon_0 c^3} r^2 \sin\theta \, d\theta \, d\phi = \frac{\omega^4 p_0^2}{16\pi\epsilon_0 c^3} \underbrace{\int_0^{\pi} \sin^3\theta \, d\theta}_{=4/3} = \frac{\omega^4 p_0^2}{12\pi\epsilon_0 c^3} \tag{5.7}$$

Hence the energy scattered by the dipole is independent of r. The amount of scattered energy is not just a function of dipole strength but is a fourth order polynomial in frequency.

5.1.2 EM Wave Approach

Consider a plane polarised wave, $E_0 e^{i\omega t}\,\hat{\mathbf{z}}$ travelling in the $\hat{\mathbf{x}}$ direction incident on a small homogeneous spherical particle. If the radius of the sphere is much smaller than the wavelength of the incident radiation then the incident electric field be considered homogeneous within the particle's volume. The applied field induces an oscillating electric dipole which produces a plane polarised electromagnetic wave (the scattered wave). This situation is shown in Figure 5.2. The electric field of the particle, caused by the electric dipole, modifies the applied field inside and near the particle. If p is the induced dipole moment then we have

$$\mathbf{p} = \alpha E_0 e^{i\omega t}\,\hat{\mathbf{z}} \tag{5.8}$$

where α is the polarizability.

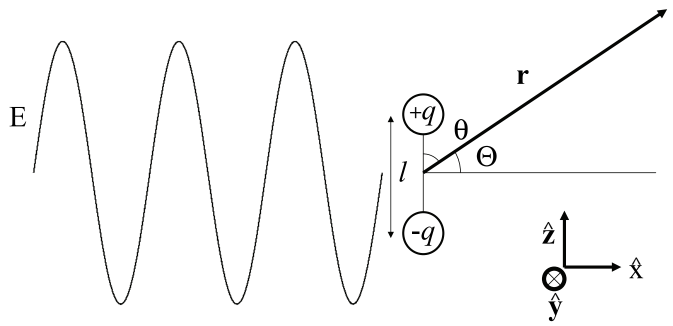

FIGURE 5.2
Scattering of linear polarised light ray. This is not to scale. The coordinate directions are shown for reference although the origin as at the centre of the dipole.

To evaluate the scattered electric field far away from the dipole let r denote the distance from the dipole to the observation point, and θ the angle between the dipole moment and the direction of observation. The plane containing the direction of propagation of the incident and scattered rays is the plane of scattering and the scattering angle, $\Theta = \pi/2 - \theta$. The electric field in the far field $\mathbf{E}_s(\mathbf{r}, t)$ is given by the expression for a Hertzian dipole (Equation 5.3),

$$\mathbf{E}_s(\mathbf{r}, t) = -\frac{k^2 p \sin\theta}{4\pi\epsilon_0 r} e^{-i\omega r/c}\,\hat{\boldsymbol{\theta}} \tag{5.9}$$

Inserting the expression for dipole strength (Equation 5.8) into this expression gives

$$\mathbf{E}_s(\mathbf{r}, t) = -\frac{k^2 \alpha \sin\theta}{4\pi\epsilon_0 r} E_0 e^{i(\omega r/c - \omega t)}\,\hat{\boldsymbol{\theta}}. \tag{5.10}$$

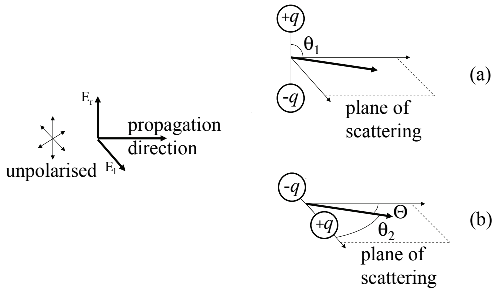

FIGURE 5.3
Scattering of unpolarised light decomposed into components (a) perpendicular to and (b) parallel to the plane of scattering.

Now consider the scattering of unpolarized light as depicted in Figure 5.3. Since any electric vector may be arbitrarily decomposed into orthogonal components we may choose these components perpendicular ($E^s_\perp$) and parallel ($E^s_\parallel$) to the plane of scattering.* Unpolarized light is characterised by the same electric field in the perpendicular and parallel directions and by a random phase relation between the two components. Thus we consider separately the scattering of the two electric field components. From Equation 5.10 we have (using $k = \omega/c$)

$$\mathbf{E}_\perp(\mathbf{r}, t) = -\frac{k^2 \alpha E_0 \sin\theta_1}{4\pi\epsilon_0 r} e^{i(\omega r/c - \omega t)}\,\hat{\boldsymbol{\theta}}. \tag{5.11}$$

$$\mathbf{E}_\parallel(\mathbf{r}, t) = -\frac{k^2 \alpha E_0 \sin\theta_2}{4\pi\epsilon_0 r} e^{i(\omega r/c - \omega t)}\,\hat{\boldsymbol{\phi}}. \tag{5.12}$$

Note that θ_1 is always $\frac{\pi}{2}$ because the dipole moment in the r direction is normal to the scattering plane. The parallel component varies with scattering angle according to $\theta_2 = \frac{\pi}{2} - \Theta$ so

$$\mathbf{E}_\perp(\mathbf{r}, t) = -\frac{k^2 \alpha E_0}{4\pi\epsilon_0 r} e^{i(\omega r/c - \omega t)}\,\hat{\boldsymbol{\theta}}. \tag{5.13}$$

$$\mathbf{E}_\parallel(\mathbf{r}, t) = -\frac{k^2 \alpha E_0 \cos\Theta}{4\pi\epsilon_0 r} e^{i(\omega r/comegat)}\,\hat{\boldsymbol{\phi}}. \tag{5.14}$$

The equivalent amplitude scattering matrix is

$$S = \begin{pmatrix} \cos\theta & 0 \\ 0 & 1 \end{pmatrix} \tag{5.15}$$

*The reference plane remains arbitrary for both forward scattering ($\Theta = 0$) and backscattering ($\Theta = 180$).

5.1.3 Radiometric Approach

Using the time averaged Poynting vector radiometric terms (Equations 3.39 and 3.40) allows Rayleigh scattering to be expressed in radiometric terms. If E^0 is the incident irradiance then the scattered power per unit steradian (i.e. the scattered intensity) components $I_\perp$ and $I_\parallel$ are

$$I_\perp = \frac{k^4\alpha^2 E^0}{16\pi^2\epsilon_0^2}\,\hat{\boldsymbol{\theta}}. \tag{5.16}$$

$$I_\parallel = \frac{k^2\alpha^2 E^0\cos^2\Theta}{16\pi^2\epsilon_0^2}\,\hat{\boldsymbol{\phi}}. \tag{5.17}$$

For unpolarized light where the energy is divided equally into the perpendicular or parallel components the scattered irradiance is the average of the components i.e.

$$I^{\mathrm{s}} = \frac{\alpha^2 k^4}{16\pi^2\epsilon_0^2}\frac{(1+\cos^2\Theta)}{2}E^0. \tag{5.18}$$

The scattering cross-section was defined in Section 3.8.1 as the equivalent area of the incident beam that intercepts the same energy as that scattered so the Rayleigh scattering cross-section is

$$\sigma^{\mathrm{sca}} = \frac{\int_0^{4\pi}\alpha^2 k^4\frac{(1+\cos^2\Theta)}{2}E^0\,d\boldsymbol{\omega}_{\mathbf{s}}}{E^0} = \frac{8}{3}\pi\alpha^2 k^4 \tag{5.19}$$

where the axes have been chosen so that $\theta = \Theta$. Using the expression for the polarizability of a homogeneous sphere, Equation 2.77 gives

$$\sigma^{\mathrm{sca}} = 32\pi^3 a^3\epsilon_0^2 k^4\left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right)^2 \tag{5.20}$$

5.1.4 Scattering from a Volume Containing Many Molecules

In the visible the refractive index is very close to 1 so that Equation 2.90 may be approximated by

$$\alpha \approx \frac{1}{4\pi N}\left(n^2 - 1\right). \tag{5.21}$$

In this case the scattering cross section σ^{sca} at wavelength λ is given by

$$\sigma^{\mathrm{sca}} = \frac{8\pi^3}{3\lambda^4 N^2}(n^2-1)^2. \tag{5.22}$$

Attenuation of light by Rayleigh scattering varies strongly with wavelength; note the λ^{-4} dependence of σ^{sca} in Equation 5.22. For a vertical column of atmosphere, 40 % of the light is scattered in the near ultraviolet while 1 % is lost in the near infrared. Rayleigh scattering explains why the sky appears blue.

The optical depth of the entire molecular atmosphere at a given wavelength, $t^{\mathrm{M}}(\lambda)$, may be calculated from the scattering cross section using:

$$t^{\mathrm{M}}(\lambda) = \sigma^{\mathrm{sca}}(\lambda) \int_0^{z_t} N(z)\, dz, \tag{5.23}$$

where $N(z)$ denotes the number density of molecules as a function of height, and z_t is the top of the atmosphere.

5.2 The Mie Solution for a Homogeneous Sphere

5.2.1 The Mie Solution

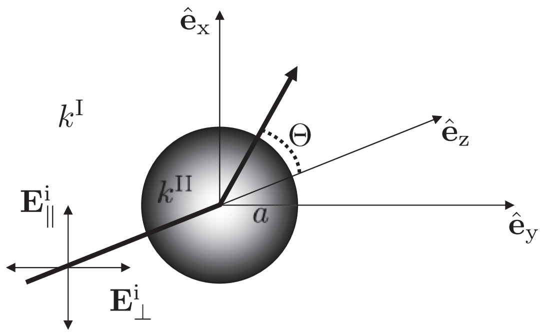

FIGURE 5.4
Geometry for Mie scattering.

Consider a homogeneous and isotropic dielectric sphere characterized by a propagation constant k^{II} (refractive index n^{II}) located at the origin and with radius a. It is surrounded by an isotropic homogeneous medium with a propagation constant k^{I} that is real (refractive index n^{I}).The relative refractive index of the sphere to the medium, $\tilde{n}$, is defined by

$$\tilde{n} = m + i\kappa = \frac{k^{\mathrm{II}}}{k^{\mathrm{I}}} = \frac{n^{\mathrm{II}}}{n^{\mathrm{I}}} \tag{5.24}$$

As before, the direction $\hat{\mathbf{e}}_x$ is chosen it so that the scattered ray travels in the x-z plane. The scattering angle is then the angle between the direction of the scattered ray and the forward direction ($\hat{\mathbf{e}}_z$). An electromagnetic plane wave, $\mathbf{E}^{\mathrm{i}}$ can then be decomposed into two linearly polarized components in the $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ direction (i.e. parallel and perpendicular to the plane of scattering respectively). This configuration is shown in Figure 5.4.

As the size of the scattering object is increased relative to the wavelength of light the phase of the incident radiation varies over the scattering object. Therefore it is convenient to introduce a dimensionless size parameter , x, which is the ratio of the circumference of the sphere to the wavelength of light, i.e.

$$x = \frac{2\pi r}{\lambda}, \tag{5.25}$$

where a is the radius of the sphere and λ the wavelength of the incident radiation. The scattering object is then approximated as an array of multipoles and the scattered wave from each of the multipoles is summed to determine the electromagnetic far field (i.e that sufficiently far from the particle that surface effects have decayed to zero [redacted] [redacted]). The scattering amplitude, as derived Appendix **??**, is

$$S_1(x;\tilde{n};\Theta) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[a_n \pi_n(\Theta) + b_n \tau_n(\Theta)\right], \tag{5.26}$$

$$S_2(x;\tilde{n};\Theta) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[a_n \tau_n(\Theta) + b_n \pi_n(\Theta)\right], \tag{5.27}$$

where

$$a_n = \frac{\psi_n(x)\psi_n'(y) - m\psi_n'(x)\psi_n(y)}{\zeta_n(x)\psi_n'(y) - m\zeta_n'(x)\psi_n(y)}, \tag{5.28}$$

$$b_n = \frac{m\psi_n(x)\psi_n'(y) - \psi_n'(x)\psi_n(y)}{m\zeta_n(x)\psi_n'(y) - \zeta_n'(x)\psi_n(y)}, \tag{5.29}$$

where ψ and ζ are Ricatti-Bessel functions (the prime indicates the derivative) and $y = \tilde{n}x$. The τ_n and π_n are defined in terms of Legendre polynomials.

The amplitudes of the components of the scattered wave are then given by

$$\begin{bmatrix} E_{\parallel}^{s} \\ E_{\perp}^{s} \end{bmatrix} = \frac{e^{ikr+ikz}}{ikr} \begin{bmatrix} S_2(\theta) & 0 \\ 0 & S_1(\theta) \end{bmatrix} \begin{bmatrix} E_{\parallel}^{i} \\ E_{\perp}^{i} \end{bmatrix} \tag{5.30}$$

where $\begin{bmatrix} S_2(\theta) & 0 \\ 0 & S_1(\theta) \end{bmatrix}$ is the *scattering amplitude matrix* . The parallel and perpendicular components of the electric field vector are indicated by the subscripts || and $\perp$ respectively.

By defining the *intensity distribution functions* , $i_1(x;\tilde{n};\Theta)$ and $i_2(x;\tilde{n};\Theta)$, as

$$i_1(x;\tilde{n};\Theta) = |S_1(\theta)|^2, \tag{5.31}$$

$$i_2(x;\tilde{n};\Theta) = |S_2(\theta)|^2. \tag{5.32}$$

the radiometric intensity of light scattered from a sphere in the far field is given by

$$I_{\parallel}^{s} = \frac{i_1}{k^2 r^2} E_{\parallel}^{i}, \tag{5.33}$$

$$I_{\perp}^{s} = \frac{i_2}{k^2 r^2} E_{\perp}^{i}, \tag{5.34}$$

where E represents the incident irradiance. ([redacted] [redacted] or transform matrix)

5.2.2 The Extinction Paradox

The efficiency factors can be calculated directly from the Mie coefficients a_n and b_n using

$$Q^{\text{sca}} = \frac{2}{x^2} \sum_{n=1}^{\infty} (2n+1) \left(|a_n|^2 + |b_n|^2 \right), \tag{5.35}$$

$$Q^{\text{ext}} = \frac{2}{x^2} \sum_{n=1}^{\infty} (2n+1) Re(a_n + b_n)). \tag{5.36}$$

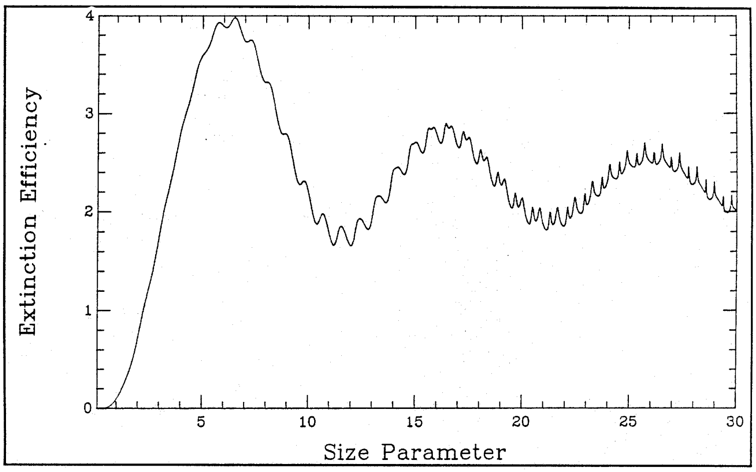

FIGURE 5.5
Extinction efficiency for a homogeneous sphere with $m = 1.33$.

As the particle becomes larger the scattering efficiency undergoes a damped oscillation about 2 and approaches 2 in the limit of large particles (see Figure 5.5). There is a rapid change from Rayleigh to Mie scattering for x changing from about 1 to 3. The efficiency factors can take on values such that the particle absorbs/scatters more flux than the geometric cross section intercepts. This means that a particle perturbs the electromagnetic field well beyond its physical confines, so the electric and magnetic fields vary continuously from the condition of the undisturbed wave far from the particle to the particle-medium boundary. Light rays are scattered from the entire region of disturbance.

5.2.3 The Mie Description of Scattered Light

When a drop is illuminated by unpolarized light of spectral irradiance, $E^i_\lambda(\theta_\mathrm{i}, \phi_\mathrm{i})$, the angular distribution of scattered light is expressed by

$$I^\mathrm{s}_\lambda(\omega_\mathrm{s}) = \frac{1}{k^2}\left[\frac{i_1(x;m;\Theta) + i_2(x;m;\Theta)}{2}\right] E^i_\lambda(\theta_\mathrm{i}, \phi_\mathrm{i}). \tag{5.37}$$

Figure 5.6 shows the increase in intensity of scattered light as the particle size is increased. For particles that are small relative to the wavelength of the scattered light the intensity changes as r^6 (Rayleigh scattering) whereas the intensity from larger particles changes as r^2. In the limit for large particles the scattered intensity can be estimated from geometric optics using the principles of diffraction, refraction and reflection.

The phase function for unpolarized light is defined as the ratio of energy scattered per unit solid angle in a given direction to the average energy scattered per unit solid

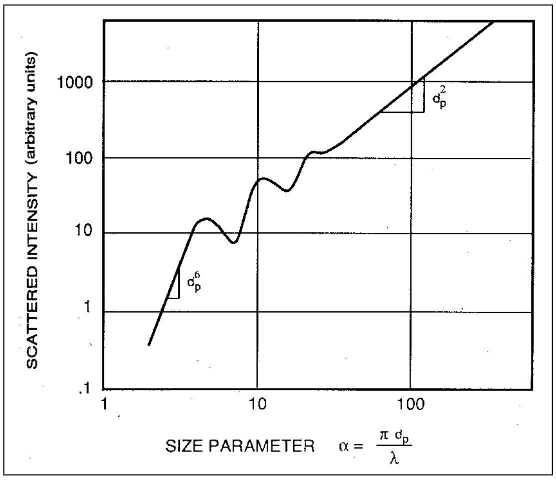

FIGURE 5.6
Scattered light as a function of size parameter.

angle in all directions

$$P(\lambda; \boldsymbol{\omega}_\mathbf{i}; \boldsymbol{\omega}_\mathbf{s}) = \frac{1}{k^2} \frac{(i_1(x; m; \Theta) + i_2(x; m; \Theta)/2}{\sigma^{sca}/4\pi}. \tag{5.38}$$

For large x the phase function has a strong diffraction peak at 0° and displays rapid variation in intensity. Neglecting the oscillations, the region from 10° to 90° is characterised by a decreasing intensity with increasing scattering angle. The primary rainbow at $\approx$ 140° arises from rays internally reflected once in spherical particles. The secondary rainbow is manifest at scattering angles just smaller than those of the primary bow: it arises from rays twice internally reflected. The supernumerary bows are the interference maxima and minima which occur at scattering angles just larger than those of the primary bow. The glory is the feature seen at 180°. The phase function changes with increasing x (see Figure 3.11) so:

- There is an increasing ratio of forward scattering to backscattering.
- The forward peak narrows and increases.
- The rainbow and glory become more pronounced.
- The minimum intensity (which occurs at $\Theta = 90°$ for Rayleigh scattering) shifts toward larger angles.
- For $x > 10$ there is a complicated dependence of scattered light intensity on the angle of observation, with the complexity increasing with x.

For very large x, when only a few scattering angles are considered, it is often convenient to use a geometric optics approach to scattering. In this case the reflection

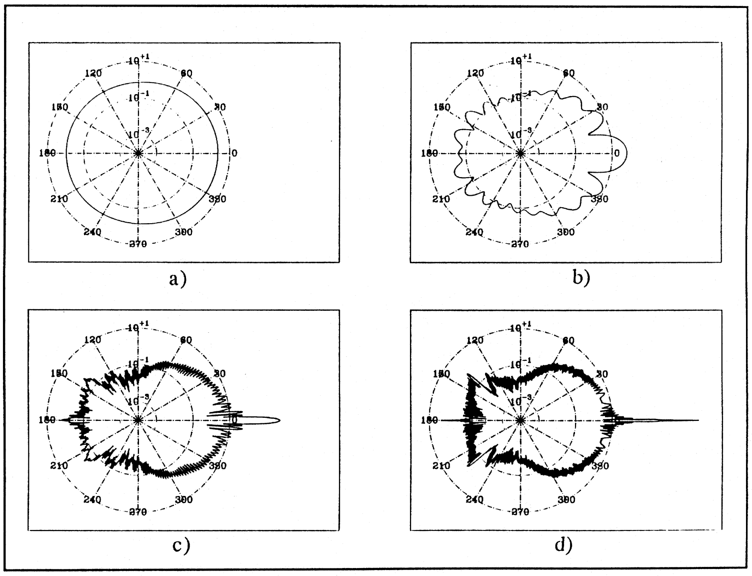

FIGURE 5.7
The phase function for water drops ($m = 1.33$): a) $x = 1$; b) $x = 10$; c) $x = 100$; d) $x = 1000$.

and refraction at the medium particle boundary is considered for a number of light rays incident on the sphere. The intensities of the scattered rays are summed (taking into account phase) and added to the sphere's Fraunhofer diffraction pattern. This technique has the advantage of being computationally fast; however, its significant drawback is that it fails for the rainbow and glory angles.

Substituting Equation 5.38 into Equation 5.37 gives

$$I^{s}_{\lambda}(\boldsymbol{\omega}_{\mathbf{s}}) = \sigma^{\text{sca}} \frac{P(\lambda; \boldsymbol{\omega}_{\mathbf{i}}; \boldsymbol{\omega}_{\mathbf{s}})}{4\pi} E^{i}_{\lambda}(\theta_{\text{i}}, \phi_{\text{i}}). \tag{5.39}$$

5.2.4 Scattering in Radiometric Terms

For a volume the redirection of irradiance $E_\lambda(\omega_i)$ from direction ω_i to direction ω_s is expressed

$$L_\lambda(\omega_s) = \beta^{\text{sca}} \frac{P(\omega_i, \omega_s)}{4\pi} E_\lambda(\omega_i) \tag{5.40}$$

which can also be expressed in terms of the incident radiance $L_\lambda(\omega_i)$ as

$$L_\lambda(\omega_s) = \beta^{\text{sca}} \frac{P(\omega_i, \omega_s)}{4\pi} L_\lambda(\omega_i)\, d\Omega_i \tag{5.41}$$

5.2.5 Scattering and Absorption by a Distribution of Particles

The Mie solution for a single sphere can be extended to give the scattering characteristics of a volume containing many spheres. The three important features of an elemental scattering volume are: the volume extinction coefficient, the single scatter albedo, and the phase function. The first of these parameters determines the depth of penetration of unscattered radiation into the medium; the second determines the relative importance of scattering to absorption; while the third gives the directional characteristic of the scattered light.

It is straightforward to extend the Mie solution for one particle to a polydispersion. We assume that the particles are sufficiently far from each other that the distance between them is much greater then the incident wavelength.

For a collection of particles, the volume coefficient or the total cross section area per unit volume is the sum of the cross section times the density of each of the species present. This is written explicitly for a continuous distribution of particle sizes as

$$\beta^x = \int_{\mathrm{r}_1}^{r_2} \sigma^x n(r)\, dr, \tag{5.42}$$

where r_1 and r_2 represent the limits of the particle size distribution and $n(r)$ denotes the number density of particles having a radius between r and $r+dr$. The superscript, x, denotes either absorption, scattering, or extinction.

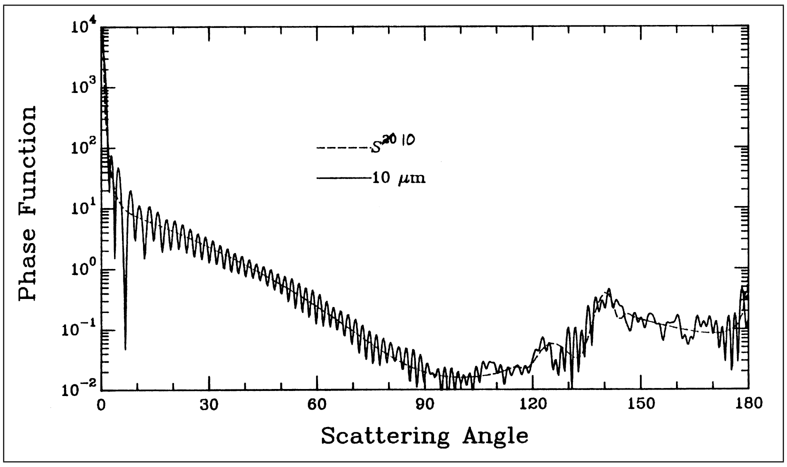

FIGURE 5.8
The phase function for a single drop and for a distribution of drops with an equivalent effective radius.

The volume phase function is expressed as

$$p(\Theta) = \frac{1}{k^2} \frac{\int_{r_1}^{r_2} [(i_1(x; m; \Theta) + i_2(x; m; \Theta))/2]\, n(r)\, dr}{\beta^{\text{sca}}/4\pi}. \tag{5.43}$$

It is evident from this equation that $p(\Theta)$ is independent of the particle concentration, that it is dimensionless, and that it meets the normalisation requirement. The integrated phase function varies smoothly compared to the rapid oscillations which occur in the phase function for a single particle (see Figure 5.8). The smooth change exhibited for a polydisperse drop distribution can be attributed to the contributions from many particles cancelling the effects that are not invariant in angle — hence features such as the forward diffraction peak and the rainbow are retained.

Problem 5.1 Derive the magnitude of the dipole moment of a sphere by comparing the second term in Equation 2.78 with the expression for the electrostatic field from a dipole given in Equation **??** in Appendix **??**.

Problem 5.2 Show that the normalised Legendre coefficients χ_l for the expansion of the Rayleigh phase function are $\{1, 0, \frac{1}{10}, 0, 0, \ldots\}$.

Hint [from *Boas*, 2006]: $\displaystyle\int_{-1}^{1} P_l(x) \cdot (\text{any polynomial of degree } < l)\, dx = 0.$

Additional Reading

Bohren, C. F., and D. R. Huffman, *Absorption and Scattering of Light by Small Particles*, Wiley, New York, 1983
Grandy, W., *Scattering of Waves from Large Spheres*, Cambridge University Press, Cambridge, 2000