7

Broadband Radiation in a Non-Scattering Atmosphere

The transfer of monochromatic radiation in the atmosphere needs to be extended to the calculation of the transfer of energy over a range of frequencies or waveband.

Two different scales of freq variation Planck + line absorption small number 20) captures variation. Small interval ? has 10^x lines economical ways of evaluating integral over frequency required.

Can assign intervals over which plank function scattering coefficient and phase function are approximately constant

i.e. want to evaluate quantity e.g. TOA irradiance as

$$E = \int_0^\infty E_{\tilde{\nu}}\, d\tilde{\nu} = \sum_{i=1}^{N} E_i \Delta\tilde{\nu}_i \tag{7.1}$$

Traditionally, the computational expense of line-by-line calculations has been avoided by using calculate the transmittance for a spectral interval where Equation 3.139

The transfer of radiation in a purely gaseous atmosphere can be determined from the description of individual molecular absorption given in Chapter 4.

7.1 Line-by-line Calculation

The monochromatic radiation radiance at height z in a non-scattering atmosphere above a black surface at temperature T_0 with no extraterrestrial sources is given by (c.f. Section 6.3.4)

$$L^{\uparrow}_{\tilde{\nu}}(z, \boldsymbol{\omega}) = B_{\tilde{\nu}}(\tilde{\nu}, T_0)\mathcal{T}(0\colon z, \mu) + \frac{1}{\mu}\int_0^z \beta^{\mathrm{abs}}(z) B_{\tilde{\nu}}(\tilde{\nu}, T(z'))\mathcal{T}(z'\colon z, \mu)\, dz' \tag{7.2}$$

$$L^{\downarrow}_{\tilde{\nu}}(z, \boldsymbol{\omega}) = \frac{1}{\mu}\int_z^{z_{\mathrm{TOA}}} \beta^{\mathrm{abs}}(z) B_{\tilde{\nu}}(\tilde{\nu}, T(z))\mathcal{T}(z\colon z', \mu)\, dz' \tag{7.3}$$

where the transmittance from height a to b is

$$\mathcal{T}(a\colon b, \mu) = \exp\left(-\frac{1}{\mu}\int_a^b \beta^{\mathrm{abs}}(z)\, dz\right). \tag{7.4}$$

Note that transmission is usually defined in terms of volume extinction coefficient but because the atmosphere is non-scattering $\beta^{\text{ext}}(z) = \beta^{\text{abs}}(z)$. The volume absorption coefficient at altitude z is a linear combination of the contribution by N absorbers so

$$\beta^{\text{abs}}(z) = \sum_{i=1}^{N} \rho_i(z) k_i^{\text{abs}}(z) \tag{7.5}$$

where $\rho_i(z)$ and $k_i^{\text{abs}}(z)$ are the absorber gas density and mass absorption coefficient of the i^{th} gas respectively. The effective mass absorption coefficient of each gas is the contribution from the M relevant lines including a continuum term if applicable i.e.

$$k_i^{\text{abs}}(p(z), T(z)) = k_{\text{continuum},i}^{abs} + \sum_{j=1}^{M} S_{ij}(z) F_{ij}(\tilde{\nu} - \tilde{\nu}_{ij}, p(z), T(z)) \tag{7.6}$$

where for the i^{th} gas, $k_{\text{continuum},i}^{abs}$ is the continuum term. The line strength, shape and location of the i^{th} line are denoted by S_{ij}, F_{ij} and $\tilde{\nu}_{ij}$ respectively.

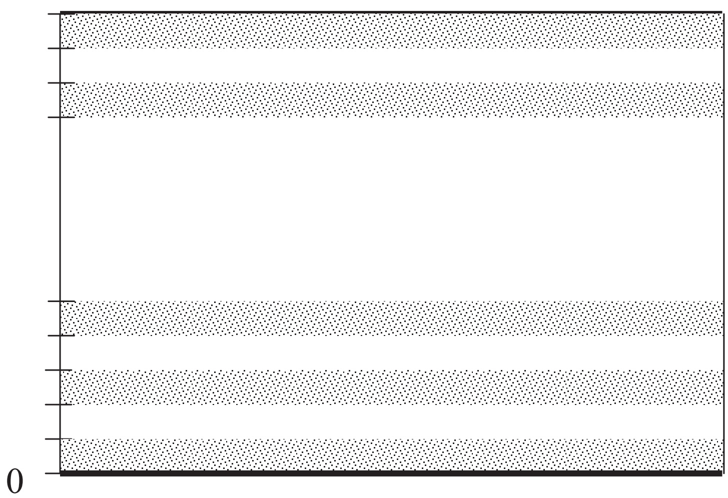

FIGURE 7.1
Computationally the atmosphere is divided into N layers ($N + 1$ levels).

The transmission through the atmosphere is built up by subdividing the atmospheric path into a number of short paths each of which is considered to be homogeneous with respect to temperature, pressure and gas concentration. This division of the atmosphere is shown in figure 7.1. The transmission through the atmosphere (from the bottom level to the top level) can be expressed

$$\mathcal{T}(0\colon z,\mu) = \prod_{n=1}^{N} \mathcal{T}_n(\mu) \tag{7.7}$$

where subscript n denotes the n^{th} atmospheric layer. The upward and downward radiance fields at the n^{th} level are then

$$L_{\tilde{\nu}}^{\uparrow}(z_n,\omega) = B_{\tilde{\nu}}(\tilde{\nu},T_0)\prod_{i=1}^{n}\mathcal{T}_i(\mu) + \frac{1}{\mu}\sum_{i=1}^{n}\beta_m^{\text{abs}}B_{\tilde{\nu}}(\tilde{\nu},T_i)\prod_{j=i+1}^{n}\mathcal{T}_j(\mu)\Delta z_j \tag{7.8}$$

$$L_{\tilde{\nu}}^{\downarrow}(z_n,\omega) = \frac{1}{\mu}\sum_{i=n+1}^{N}\beta_m^{\text{abs}}B_{\tilde{\nu}}(\tilde{\nu},T_i)\prod_{j=n+1}^{i-1}\mathcal{T}_j(\mu)\Delta z_j \tag{7.9}$$

where T_0 is the surface temperature and Δz_j is the thickness of the j^{th} layer.

The transmission through each of these path segments is calculated from knowledge of the trace gas amounts and the line strength, centre, and width of each component gas. This is a computationally intense calculation as it is not just lines within the spectral range of interest that need to be considered. The contribution from the wings of many absorption lines centred far from the spectral region of interest can sum to produce significant absorption within the spectral band. Once the transmissions are known they can be used in Equations 7.8 or 7.9 to numerically estimate the radiance field at a given altitude in a specific direction.

7.2 Band Transmission

The estimation of radiative energy flows within the atmosphere requires the integration of the spectral radiance field over a range of frequencies centred at $\tilde{\nu}_0$ say. If the interval is sufficiently small that the Planck function is approximately constant the expressions for the mean radiance are the same as those given in Equations 7.8 and 7.9 with the transmittance (Equation 7.4) replaced by the average transmittance over the frequency interval, the band transmittance, defined by

$$\overline{\mathcal{T}}_l(\tilde{\nu}_0,\Delta\tilde{\nu},\mu) = \frac{1}{\Delta\tilde{\nu}}\int_{\tilde{\nu}_0-\Delta\tilde{\nu}/2}^{\tilde{\nu}_0+\Delta\tilde{\nu}/2}\exp\left(-\frac{1}{\mu}\int_a^b\beta^{\text{abs}}(z)\,dz\right)d\tilde{\nu} \tag{7.10}$$

so the problem of integrating the entire expression for radiance simplifies into an integration to generate the transmittance.

Similarly the optical depth of the band is

$$\overline{\tau}_l(\tilde{\nu}_0,\Delta\tilde{\nu},\mu) = -\ln\overline{\mathcal{T}}_l(\tilde{\nu}_0,\Delta\tilde{\nu},\mu) = \frac{1}{\Delta\tilde{\nu}}\int_{\tilde{\nu}_0-\Delta\tilde{\nu}/2}^{\tilde{\nu}_0+\Delta\tilde{\nu}/2}\left(\frac{1}{\mu}\int_a^b\beta^{\text{abs}}(z)\,dz\right)d\tilde{\nu} \tag{7.11}$$

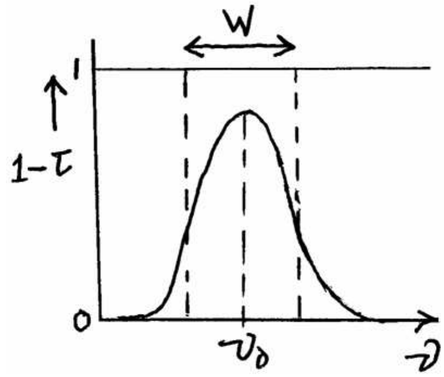

FIGURE 7.2
Equivalent width of a line.

and the band averaged absorptance is

$$\overline{\mathcal{A}} = 1 - \overline{\mathcal{T}} \tag{7.12}$$

7.2.1 Equivalent Width

Now consider a path through a gas where the conditions along the path are homogeneous. The integrated absorption of a spectral line, W, belonging to the gas is defined as

$$W = \int_0^\infty [1 - \mathcal{T}(\tilde{\nu})]\, d\tilde{\nu}. \tag{7.13}$$

This expression defines the *equivalent width*, W, which is the width of a rectangular line of 100 % absorption with the same area as the absorption line. This is shown in Figure 7.2. Here equivalent width is expressed in terms of wavenumber but a similar definition can be made in terms of wavelength. The equivalent width and is proportional to the energy absorbed by an isolated line. As equivalent width is only weakly dependent upon instrument line shape it is a useful metric in comparing measurements where the shape of the spectral line is not resolved. Equation 7.13 is sometimes expressed with respect to the line centre or the bottom limit is changed to $-\infty$. This latter modification simplifies the evaluation of analytic line shapes and is possible because of the negligible contribution to the integral of negative frequencies. The band-average absorptance is

$$\overline{\mathcal{A}} = \frac{W}{\Delta\tilde{\nu}} \tag{7.14}$$

The transmittance is related to the mass per unit area m of the absorbing gas through

$$\mathcal{T}(\tilde{\nu}) = \exp(-k^{\text{abs}}(\tilde{\nu})m) \tag{7.15}$$

so that the equivalent width as a function of m is

$$W = \int_0^\infty [1 - \exp(-k^{\rm abs}(\tilde{\nu})m)]\, d\tilde{\nu}. \tag{7.16}$$

To understand how W changes with m it is useful to examine the two limiting cases.

7.2.2 Weak Approximation

If the line absorption is weak then Equation 7.15 becomes

$$\mathcal{T} \sim 1 - k^{\rm abs}(\tilde{\nu})m$$

so that

$$\begin{aligned} W &= \int_0^\infty k^{\rm abs}(\tilde{\nu})m\, d\tilde{\nu} \\ &= m \int_0^\infty k^{\rm abs}(\tilde{\nu})\, d\tilde{\nu} \\ &= S m \end{aligned}$$

where $S = \int_0^\infty k^{\rm abs}(\tilde{\nu})\, d\tilde{\nu}$ is the line strength. This is called the weak approximation and is true for any line shape. Note that $W \propto m$.

7.2.3 Strong Approximation

In the case of a pressure-broadened line there is also a *strong* approximation which corresponds to complete absorption in the line centre and any increase in m (or p) only affects the wings. For a Lorentz line:

$$k^{\rm abs}(\tilde{\nu}) = \frac{S}{\pi} \frac{\alpha_{\rm L}}{((\tilde{\nu} - \tilde{\nu}_0)^2 + \alpha_{\rm L}^2)},$$

so

$$W = \int_{-\infty}^\infty \left[1 - \exp\left(-\frac{S}{\pi} \frac{\alpha_{\rm L}}{((\tilde{\nu} - \tilde{\nu}_0)^2 + \alpha_{\rm L}^2)} m\right)\right] d\tilde{\nu}. \tag{7.17}$$

Note that the lower limit of this integral has been set at $-\infty$ in order to determine an analytic solution of the integral.

For a strong line radiation near the line centre is completely absorbed and the value of $k^{\rm abs}(\tilde{\nu})$ near the centre is not important. At some distance from the line centre the α_L^2 can be ignored in the denominator so

$$W = \int_0^\infty \left[1 - \exp\left(-\frac{S\,\alpha_{\rm L} m}{\pi(\tilde{\nu} - \tilde{\nu}_0)^2}\right)\right] d\tilde{\nu}$$

Then setting $x^2 = (\tilde{\nu} - \tilde{\nu}_0)^2$ gives

$$W = \int_{-\infty}^{\infty} \left[1 - \exp\left(-\frac{S\,\alpha_{\rm L} m}{\pi x^2}\right)\right] dx.$$

Putting $\beta = S\,\alpha_{\rm L} m/\pi x^2$ gives

$$\frac{d\beta}{dx} = -2\frac{S\,\alpha_{\rm L} m}{\pi x^3} \Rightarrow \frac{dx}{d\beta} = -\frac{1}{2\beta^{3/2}}\sqrt{\frac{S\,\alpha_{\rm C} m}{\pi}}.$$

Hence

$$\begin{aligned} W &= 2\int_{0}^{\infty} \left[1 - \exp\left(-\frac{S\,\alpha_{\rm L} m}{\pi x^2}\right)\right] dx, \\ &= \sqrt{\frac{S\,\alpha_{\rm L} m}{\pi}} \int_{0}^{\infty} \frac{1 - \exp(-\beta)}{\beta^{3/2}}\, d\beta. \end{aligned}$$

The integral can be evaluated to $2\sqrt{\pi}$. Thus

$$W = 2\sqrt{S\,\alpha_{\rm L} m}.$$

7.2.4 Curve of Growth

For a pressure broadened line the general solution for the equivalent width is found by substituting $m' = S\,m/2\pi\alpha_L$ into Equation 7.17 to get

$$W = \int_{-\infty}^{\infty} \left[1 - \exp\left(-\frac{2\alpha_{\rm L}^2 m'}{((\tilde{\nu} - \tilde{\nu}_0)^2 + \alpha_{\rm L}^2)}\right)\right] d\tilde{\nu}. \qquad (7.18)$$

which can be solved analytically (see Problem 7.2) to give

$$W = 2\pi\alpha_{\rm L} m' e^{-m'} [I_0(m') + I_1(m')]$$

where $I_n(x) = i^{-n} J_n(ix)$ are modified Bessel functions of the first kind of order n and J_n is the nth order Bessel function.

Figure 7.3 shows the equivalent width, W, plotted against absorber density, m. This figure is referred to as the curve of growth. From this the weak and strong limits can be seen.

7.2.5 Band Models

In a band model we use a statistical description of the distribution and strength of lines to estimate the band transmittance. The quantities we are concerned with are:

- The distribution of line position within the band. The two usual assumptions are either random or regular.

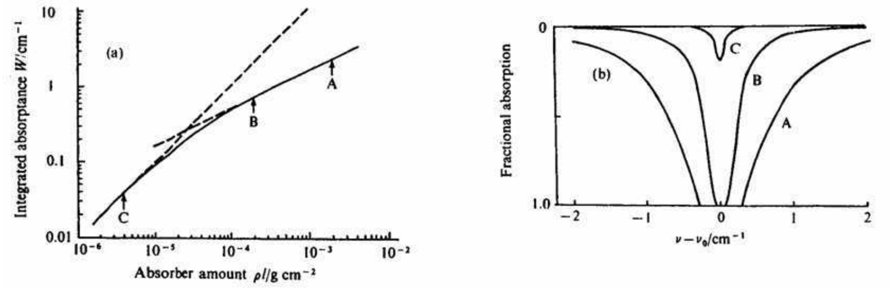

FIGURE 7.3
Need to redo this figure it uses $S = 10^{-4}(\mathrm{g\ cm}^{-2})^{-1}$, $\alpha_0 = 0.06\ \mathrm{cm}^{-1}$

- The statistical distribution function, $p(S)$, of line strengths. Common assumptions are:

δ-distribution	$p(S) = A\delta(S - \overline{S})$
Exponential distribution	$p(S) = A/\overline{S}\exp(-S/\overline{S})$
Malkmus distribution	$p(S) = A/S\exp(-S/\overline{S})$

  where A is a normalization constant chosen so that

$$\int_0^\infty p(s)\,dS = 1 \tag{7.19}$$

  and $\overline{S}$ is the mean line strength given by

$$\overline{S} = \int_0^\infty S\,p(s)\,dS \tag{7.20}$$

- The line widths - typically the widths of all lines within the band are assumed to be the same.

For example if we have a spectral interval $\Delta\tilde{\nu}$ centred at $\tilde{\nu}_0$ that contains N lines then the mean line spacing $\overline{\delta}$ is

$$\overline{\delta} = \frac{\Delta\tilde{\nu}}{N}.$$

If the lines do not overlap then we can calculate a total equivalent width for the interval as

$$W = \sum_1^N W_i$$

and an effective mean transmission $\overline{\mathcal{T}}(\tilde{\nu}_0, \Delta\tilde{\nu})$ as

$$\begin{aligned}\overline{\mathcal{T}}(\tilde{\nu}_0, \Delta\tilde{\nu}) &= 1 - \frac{W}{\Delta\tilde{\nu}} \\ &= 1 - \frac{1}{\overline{\delta}}\frac{\sum_1^N W_i}{N}.\end{aligned}$$

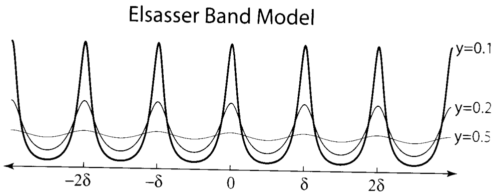

FIGURE 7.4
Need to redo this figure

If the lines are 'weak' (so $W_i = S_i m$) then

$$\overline{\mathcal{T}}(\tilde{\nu}_0, \Delta\tilde{\nu}) = 1 - \frac{1}{\delta}\frac{\sum_1^N S_i}{N} m = 1 - \frac{1}{\overline{\delta}}\overline{S} m$$

where $\overline{\delta}$ and $\overline{S}$ are the mean line spacing and mean line strength for the interval $\Delta\tilde{\nu}$. If the spectrum is divided into a series of such intervals, $\overline{\delta}$ and $\overline{S}$ contain the spectral information as slowly varying functions of $\tilde{\nu}$. This is the basic idea of the *transmission function*. The $\overline{\delta}$ and $\overline{S}$ can be precomputed from spectral data so that the integration over frequency is reduced to the summation over a relatively small number of spectral intervals.

The result for non-overlapping 'strong' lines ($W_i = 2\sqrt{S_i \alpha_{L_i} m}$) is

$$\overline{\mathcal{T}}(\tilde{\nu}_0, \Delta\tilde{\nu}) = 1 - \frac{1}{\delta}\frac{\sum_1^N \sqrt{S_i \alpha_{L_i}}}{N} 2\sqrt{m}.$$

In general, lines are not independent (they overlap), nor are they 'weak'. If an empirical model of a spectral band is used then it is possible to derive transmission functions in the case of overlapping lines.

7.2.6 The Elsasser Regular Model

The regular band model was developed by *Elsasser* [1938]. A spectrum of equally spaced (separation δ) identical Lorentz lines of strength S is assumed so that the absorption coefficient at a wavenumber displaced $\tilde{\nu}$ from the centre of a line is

$$k^{\rm abs}(\tilde{\nu}) = \sum_{i=-\infty}^{\infty} \frac{S}{\pi} \frac{\alpha_{\rm L}}{((\tilde{\nu} - i\delta)^2 + \alpha_{\rm L}^2)},$$

This is equivalent [*Elsasser*, 1938] to

$$k^{\rm abs}(\tilde{\nu}) = \sum_{i=-\infty}^{\infty} \frac{S}{\delta} \frac{\sinh\beta}{\cosh\beta - \cos\gamma},$$

where $\beta = 2\pi\alpha_{\rm L}/\delta$ and $\gamma = 2\pi\tilde{\nu}/\delta$. Figure 7.4 shows that if β is large then the adjacent lines overlap while for small β the lines appear isolated. The band transmittance can be shown [*Liou*, 1980] to be

$$\overline{\mathcal{T}}(\tilde{\nu}_0, \Delta\tilde{\nu}) = \int_0^\infty e^{-z\coth\beta} J_0(iz/\sinh\beta)$$

where $z = y\sinh\beta$. This function must be evaluated numerically however when β is large the line width is much greater than the spacing between lines so that absorption approaches a continuum. In this case the band transmittance is [*Petty*, 2004]

$$\overline{\mathcal{T}}(\tilde{\nu}_0, \Delta\tilde{\nu}) = e^{-\frac{S}{\delta}m}$$

so that medium acts like it has a mass absorption coefficient $k^{\rm abs} = \frac{S}{\delta}$. For small values of $\sqrt{Sm\pi\alpha_{\rm L}}/\delta$ the band transmittance is [*Petty*, 2004]

$$\overline{\mathcal{T}}(\tilde{\nu}_0, \Delta\tilde{\nu}) = 1 - {\rm erf}\left[\frac{1}{\delta}\sqrt{Sm\pi\alpha_L}\right].$$

where erf is the error function.

7.2.7 The Goody Random Model

The Goody random model [*Goody*, 1952, 1964] assumes a spectrum of lines which are random in position and strength. $\Delta\tilde{\nu}$ spectral interval containing N lines of mean separation δ. If the strength S_i $i^{\rm th}$ line is described by probability distribution function $p(S_i)$ then the expected band transmittance from a single line is

$$\overline{\mathcal{T}}_i(\tilde{\nu}_0, \Delta\tilde{\nu}) = \frac{1}{\Delta\tilde{\nu}} \int_{\tilde{\nu}_0-\Delta\tilde{\nu}/2}^{\tilde{\nu}_0+\Delta\tilde{\nu}/2} \int_0^\infty p(S_i) e^{k_i^{\rm abs} m}\, dS_i\, d\tilde{\nu} \tag{7.21}$$

where $p(S_i)$ is fulfils the normalization requirement of Equation 7.19. Then the product of the transmittance from N lines gives the expect transmittance i.e.

$$\begin{aligned}\overline{\mathcal{T}}(\tilde{\nu}_0, \Delta\tilde{\nu}) = & \frac{1}{\Delta\tilde{\nu}} \int_{\tilde{\nu}_0-\Delta\tilde{\nu}/2}^{\tilde{\nu}_0+\Delta\tilde{\nu}/2} \int_0^\infty p(S_1) e^{k_1^{\rm abs} m}\, dS_1\, d\tilde{\nu} \times \\ & \frac{1}{\Delta\tilde{\nu}} \int_{\tilde{\nu}_0-\Delta\tilde{\nu}/2}^{\tilde{\nu}_0+\Delta\tilde{\nu}/2} \int_0^\infty p(S_2) e^{k_2^{\rm abs} m}\, dS_2\, d\tilde{\nu} \times \ldots\end{aligned} \tag{7.22}$$

All these integrals are the same so

$$\overline{\mathcal{T}}(\tilde{\nu}_0, \Delta\tilde{\nu}) = \left[\frac{1}{\Delta\tilde{\nu}} \int_{\tilde{\nu}_0-\Delta\tilde{\nu}/2}^{\tilde{\nu}_0+\Delta\tilde{\nu}/2} \int_0^\infty p(S) e^{k^{\rm abs} m}\, dS\; d\tilde{\nu}\right]^N \tag{7.23}$$

$$= \left[\frac{1}{\Delta\tilde{\nu}} \int_{\tilde{\nu}_0-\Delta\tilde{\nu}/2}^{\tilde{\nu}_0+\Delta\tilde{\nu}/2} \int_0^\infty p(S)(1 - 1 + e^{k^{\rm abs} m})\, dS\; d\tilde{\nu}\right]^N \tag{7.24}$$

$$= \left[1 - \frac{1}{\Delta\tilde{\nu}} \int_{\tilde{\nu}_0-\Delta\tilde{\nu}/2}^{\tilde{\nu}_0+\Delta\tilde{\nu}/2} \int_0^\infty p(S)(1 - e^{k^{\rm abs} m})\, dS\; d\tilde{\nu}\right]^N \tag{7.25}$$

(7.26)

The equivalent width for N absorption line is

$$W = \frac{1}{\Delta\tilde{\nu}} \int_{\tilde{\nu}_0-\Delta\tilde{\nu}/2}^{\tilde{\nu}_0+\Delta\tilde{\nu}/2} \int_0^\infty p(S)(1 - e^{k^{\text{abs}} m})\, dS\, d\tilde{\nu} \tag{7.27}$$

So that

$$\overline{\mathcal{T}}(\tilde{\nu}_0, \Delta\tilde{\nu}) = \left[1 - \frac{1}{N}\frac{W}{\delta}\right]^N \tag{7.28}$$

The Goody model assumes that the line strengths have a probability distribution of

$$P(S) = \frac{1}{S}\exp(-S/\bar{S})$$

so

$$\int_0^\infty p(S)(1 - e^{k^{\text{abs}} m})\, dS = \int_0^\infty \frac{1}{S}\exp(-S/\bar{S})(1 - e^{k^{\text{abs}} m})\, dS \tag{7.29}$$

$$= \int_0^\infty \frac{1}{S}\exp(-S/\bar{S})\, dS - \int_0^\infty \frac{1}{S}\exp(-S/\bar{S}) e^{S f m}\, dS \tag{7.30}$$

$$= 1 - \int_0^\infty \frac{1}{S}\exp(-S(fm + \bar{S}))\, dS \tag{7.31}$$

$$= 1 - \left[(1 - \bar{S} fm)\exp(-S(fm + 1/\bar{S}))\right]_0^\infty \tag{7.32}$$

$$= \frac{\bar{S} fm}{1 + \bar{S} fm} \tag{7.33}$$

Using the Lorenz line shape (Equation 4.37) but performing the integral from $-\infty$ to ∞ gives The equivalent width for N absorption line is

$$W = \frac{1}{\Delta\tilde{\nu}} \int_{-\infty}^{\infty} \frac{\bar{S} fm}{1 + \bar{S} fm}\, d\tilde{\nu} \tag{7.34}$$

$$= \frac{1}{\Delta\tilde{\nu}} \int_{-\infty}^{\infty} \frac{\bar{S} m \alpha_C}{\pi((\nu - \nu_0)^2 + \alpha_C^2 + \bar{S} m \alpha_C}\, d\tilde{\nu} \tag{7.35}$$

$$= \bar{S} m \sqrt{1 + \frac{\bar{S} m}{\pi\bar{\alpha}}} \tag{7.36}$$

which gives

$$\overline{\mathcal{T}} = \exp\left[-\frac{\bar{S} m}{\delta}\sqrt{1 + \frac{\bar{S} m}{\pi\bar{\alpha}}}.\right]$$

7.2.8 Curtis-Godson Approximation

The transmission functions discussed so far apply to a uniform path. In the atmosphere the absorption coefficient depends on pressure and temperature, and these, as well as the absorber concentration, vary with height, sometimes over several orders of magnitude.

A ray passing through a non-homogeneous atmosphere forms an absorption line from the superposition of many Lorentz profiles which is inconsistent with any one Lorentz profile.

The Curtis-Godson approximation allows us to estimate the pressure, temperature and absorber amount for an equivalent uniform path so that a single Lorentz line corresponds as well as possible with the superimposed profile.

$$\bar{p} = \frac{1}{m}\int p\rho\, dz \quad \bar{T} = \frac{1}{m}\int T\rho\, dz$$

where

$$m = \int \rho\, dz$$

is the total mass per unit cross-sectional area of the path.

For a gas (e.g. CO_2) with a constant mixing ratio, χ, in an atmosphere in hydrostatic equilibrium then

$$dp = -g\rho_a\, dz \quad \text{and} \quad \chi = \rho/\rho_a \quad \text{so} \quad \rho\, dz = -\frac{\chi}{g}\, dp,$$

so

$$\begin{aligned}
\bar{p} &= \frac{\int_{p_1}^{p_2} p\frac{\chi}{g}\, dp}{\int_{p_1}^{p_2} \frac{\chi}{g}\, dp}, \\
&= \frac{1}{2}\frac{(p_2^2 - p_1^2)}{(p_1 - p_2)}, \\
&= \frac{1}{2}(p_1 + p_2), \\
&= \text{mean pressure.}
\end{aligned}$$

In general, χ is not constant although it is a useful approximation over a relatively thin layer.

In the case of Lorentz broadened lines in the strong limit we may write for the transmission of a path from z_1 to z_2 at frequency ν

$$\tau = \exp\left[-\int_{z_1}^{z_2} \frac{S}{\pi}\frac{\alpha_L}{(\nu-\nu_0)^2}\rho\, dz\right]$$

where $\alpha_L = \alpha_0\left(\frac{p}{p_0}\right)\left(\frac{T_0}{T}\right)^{\frac{1}{2}} \sim \alpha_0 p/p_0$.

$$\begin{aligned}
\tau &= \exp\left[-\frac{S}{\pi}\frac{\alpha_0}{(\nu-\nu_0)^2}\frac{1}{p_0}\int_{z_1}^{z_2} p\rho\, dz\right] \\
&= \exp\left[-\frac{S}{\pi}\frac{\alpha_0}{(\nu-\nu_0)^2}\frac{\bar{p}m}{p_0}\right]
\end{aligned}$$

where $\bar{p} = \frac{1}{m}\int p\rho\, dz$ is the Curtis-Godson mean pressure. Thus the approximation is exact in the 'strong' limit. It is also exact in the 'weak' limit when the equivalent width is independent of pressure. (Show this).

7.3 Heating Rates

$$-\frac{dE^{\uparrow}_{\tilde{\nu}}}{dt^*} + E^{\uparrow}_{\tilde{\nu}} = \pi B_{\tilde{\nu}}$$

If we integrate the radiative transfer equation over frequency assuming the atmosphere is grey (i.e. non-scattering with an extinction coefficient that is independent of frequency) we obtain

$$\begin{aligned} -\frac{dF^{\uparrow}}{dt^*} + F^{\uparrow} &= \pi B, \\ \frac{dF^{\downarrow}}{dt^*} + F^{\downarrow} &= \pi B. \end{aligned}$$

This is called the *two-stream approximation*. Note that B is the spectrally integrated Planck function so that $\pi B(T) = \sigma T^4$. In each of these equations the first term represents the rate of change of the irradiance along the path, the second term represents the extinction and the third term represents the emission which happens equally in both directions. Remember that irradiance has a specific direction associated with it. If the surface in question is horizontal and the normal, $\vec{\mathbf{n}}$, points upwards then the irradiance under consideration, $F^{\uparrow}$, is associated with upward-moving photons. The net flux, $\vec{\mathbf{F}}$, is a vector quantity that represents in magnitude and direction the net irradiance at a given point in space.

$$\vec{\mathbf{F}} = (F^{\uparrow} - F^{\downarrow})\vec{\mathbf{n}}$$

where $F^{\downarrow}$ is the irradiance from downward-moving photons.

The heating rate in the atmosphere depends on the divergence of $\vec{\mathbf{F}}$. It represents the net power removed from the medium per unit volume as a result of the radiation stream.

$$\begin{aligned} \frac{dT}{dt}\rho_a C_p &= -\nabla \cdot \vec{\mathbf{F}} \\ &= -\frac{d}{dz}(F^{\uparrow} - F^{\downarrow}) \end{aligned}$$

where $F^{\uparrow}$ is measured in the same direction as z and ρ_a is the density of the air. The heating rate $\frac{dT}{dt}$ is usually expressed in K day^{-1}.

Given that the atmosphere is transparent in the short-wave, there can be no short-wave heating. If we assume that the atmosphere is in *radiative equilibrium* then the long wave heating is also zero. Hence the net upward long-wave radiation F is independent of height

$$F = F^{\uparrow} - F^{\downarrow} = \text{ constant (net upward irradiance).}$$

Let the total irradiance, ξ, be defined

$$\xi = F^{\uparrow} + F^{\downarrow}.$$

Then on substituting the diffuse approximation equations we get

$$\frac{dF}{dt^*} = \frac{dF^{\uparrow}}{dt^*} - \frac{dF^{\downarrow}}{dt^*} = (F^{\uparrow} - \pi B) - (-F^{\downarrow} + \pi B) = \xi - 2\pi B,$$
$$\frac{d\xi}{dt^*} = \frac{dF^{\uparrow}}{dt^*} + \frac{dF^{\downarrow}}{dt^*} = (-F^{\downarrow} + \pi B) + (F^{\uparrow} - \pi B) = F.$$

However we know F =constant so $\frac{dF}{dt^*} = 0$. Thus

$$\xi = 2\pi B \text{ and } F = 2\pi \frac{dB}{dt^*},$$

which integrates to give

$$B = \frac{1}{2\pi} F t^* + \text{constant.}$$

Using the boundary condition that $F^{\downarrow} = 0$ at $t^* = 0$ gives

$$B = \frac{1}{2\pi} F(1 + t^*).$$

Note that $F^{\uparrow}$ at $t^* = 0$ must balance the incoming unreflected short-wave irradiance which we take as $\approx 240\,\mathrm{W\,m^{-2}}$.

Note that there is a discontinuity between the ground temperature T_g and the temperature near the ground $T(t_g^*)$ The two are related by

$$\pi B(T(t_g^*)) = \pi B(T_g) - \frac{1}{2} F \text{ also } \pi B(T(t_g^*)) = \frac{1}{2} F(1 + t_g^*).$$

Eliminating $B(t_g^*)$ (the air temperature near the ground) from these two equations gives the surface temperature

$$\pi B(T_g) = \frac{1}{2} F(2 + t_g^*).$$

Thus if the optical depth of the whole atmosphere, t^*, is non-zero the emission from the surface will be greater than the outgoing emission at the top of the atmosphere (σT_g^4 is larger than $F^{\uparrow}(0)$). Thus the presence of the atmosphere increases the surface temperature. The greenhouse effect again!

Problem 7.1 A cylindrical cell of radius 5 cm and length 1 m contains a single gas. A transmission measurement of 0.27 is made in a waveband where all the lines are isolated and at the weak limit with strength, S = . What is the amount of gas in the cell?

Problem 7.2 Show that the solution of Equation 7.18 is

$$L(m') = m' \exp(-m')[J_0(im') - iJ_1(im')].$$

where J_0 and J_1 are the modified Bessel functions of the first kind.

Additional Reading

Wolfe, W. L., *Introduction to Radiometry*, SPIE Optical Engineering Press, 1998.