Prelims: Mechanics II

1. (a) Show that in an elastic collision of two particles masses m_1 and m_2 viewed in the centre-of-mass frame, the magnitude of the velocity of each particle is unchanged by the collision.

   (b) A particle P_1 mass m with vector velocity $\mathbf{u}_1$ collides elastically with an identical particle P_2 also of mass m that is at rest. The velocities of the particles after the collision are $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Working in the laboratory frame, show that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ and hence deduce that, after the collision, either (i) P_1 comes to rest or (ii) P_1 and P_2 move at right angles. Show also that $\mathbf{u}_1 \cdot \mathbf{v}_1 > 0$, and hence deduce that P_1 is scattered through an angle of less than 90°wrt the incident particle.

   (c) Solve the problem in (b) by considering the same collision as viewed in the centre-of-mass frame and then transforming to the lab.

2. An alpha particle of mass $4m$ moves at non-relativistic speed and collides elastically and obliquely with a stationary proton, mass m. Draw diagrams illustrating the situation after the collision in the centre-of-mass frame and in the lab frame. Using a geometrical or algebraic method, show that in the lab frame the alpha particle cannot be deflected by more than 14.5° from its original direction of motion.

3. A particle of mass $2m$ travelling at a non-relativistic speed u collides inelastically with a stationary particle of mass m. By considering the collision in the centre-of-mass frame, show that the speed in the laboratory frame of the less massive particle, after the collision, is

$$v_2 = \frac{2u}{3}\left(\cos\theta \pm \sqrt{\alpha^2 - sin^2\theta}\right),$$

   where θ is the angle between the particle's trajectory and the original direction of the more massive particle, and α is the coefficient of restitution, defined as the magnitude of the relative velocity of the particles after the collision divided by the magnitude of the relative velocity before the collision.

4. (a) Show that the work done by the force $F(x)$ to move a particle of mass m from a to b is given by: $W_{ab} = T(b) - T(a)$ where $T(x)$ is the kinetic energy at the point x. Does this only apply for conservative forces?

   (b) The particle moves towards the origin of the x-axis under the influence of the force $F = k/x^2$ where k is a positive constant. If the velocity at $x = x_0$ is u, use the result in (a) to find the distance of minimum approach of the particle to the origin.

5. A particle P of mass m moves on the x-axis under the combined gravitational attraction of two particle, each of mass M, fixed at the points $(0, \pm a)$ in the $x - y$ plane.

   (a) Show that the force acting on the particle P is given by

$$F(x) \quad = \quad -\frac{2mMGx}{(a^2 + x^2)^{3/2}}$$

   where G is the gravitation constant.

(b) Find the corresponding potential energy. Discuss the possible motion of the particle using the potential energy curve.

(c) Assuming that P is initially released from rest at $x = 3a/4$, find the maximum speed achieved by the particle in the subsequent motion.

6. The Lennard-Jones potential describes the potential energy between two atoms in a molecule

$$U(r) = \epsilon[(r_0/r)^{12} - 2(r_0/r)^6],$$

where ϵ and r_0 are constants and r is the distance between the atoms.

(a) Sketch $U(r)$ and find the position of the minimum potential energy and the depth of the potential well (this should identify the constants).

(b) Expand $U(r)$ as a Taylor series about $r = r_0$ up to the quadratic term.

(c) Use (b) to show that the motion for small displacements about the minimum is simple harmonic and find its frequency (remember to use the reduced mass). [Ans: $\omega = (12/r_0)\sqrt{\epsilon/m}$.]

(d) Typical vibrational frequencies of diatomic molecules lie in the near infrared (around 3×10^{13} Hz). Estimate the value of the effective spring constant for a typical molecule, such as N_2.

7. A ball is thrown with initial speed V, at an angle θ above the horizontal, over flat ground. Neglecting air resistance, find

(a) the time taken for the ball to reach the ground;

(b) the maximum height reached by the ball;

(c) the horizontal distance travelled by the ball before hitting the ground (range);

(d) the kinetic and potential energies at the position of maximum height (and verify that their sum equals the initial energy).

The following are experimental data on the range and muzzle velocity of mortar shells, all fired at $45°$ to the horizontal. The time of flight is also tabulated. Compare the ranges and times with the simple theory above. How would you explain any differences?

Muzzle Velocity (m/s)	Range (m)	Time (s)
101.8	972	14.4
112.2	1160	15.7
121.9	1349	17.0
131.4	1539	18.2

8. A body of mass 1 kg falls from a height of 100 m and reaches the ground with a speed of $40\,\mathrm{m\,s^{-1}}$. What is the average air resistance? Discuss the meaning of the term 'average' in this connection.

9. An air-filled toy balloon, with a diameter of 30 cm and a mass (not counting the air inside) of about 0.5 g, falls from rest in air.

   (a) Estimate the terminal speed of fall, assuming a linear drag law [i.e. assume the balloon is a sphere, and that Stokes' formula for viscous drag is applicable: drag force $F_1 = 6\pi a\eta v$[N], where a is the radius of the sphere [m], $\eta = 1.7 \times 10^{-5}\,\mathrm{kg\,m^{-1}\,s^{-1}}$ is the viscosity of air and v is velocity].

   (b) Given this drag law, show that the downward velocity v of the balloon satisfies a differential equation of the form

$$\frac{dv}{dt} = g - \lambda v$$

   and give an expression for the quantity λ. Solve this differential equation, using the appropriate initial condition, and hence find how long it takes for the balloon to come to within 5 per cent of its terminal speed.

   (c) How would your result for (a) change if an additional drag force due to turbulent eddies of the form $F_2 = 0.87(av)^2$ [N] (where a is in m and v is in $\mathrm{m\,s^{-1}}$) were taken into account? Which is more realistic?

10. A ball of mass m is projected vertically upwards at a velocity v_0. The ball experiences an air resistance force (in addition to gravity) of the form $F_R = -\alpha v^2$ where $\alpha > 0$ is constant and v is the velocity, and reaches a maximum height h before it returns back to the point of projection.

   (a) Write down the equations of motion of the ball during its upward and and downward journeys and show that the maximum height reached is given by

$$h = a\ln[1 + (v_0/v_l)^2]$$

   where $v_l = \sqrt{mg/\alpha}$ and $a = v_l^2/(2g)$.

   (b) Show that the velocity of the ball when it returns back to the point of projection is given by

$$v_r^2 = v_l^2[1 - \exp(h/a)]$$

11. Denote the radius of the Earth by R_{E} and the value of the acceleration due to gravity at the surface by g.

   (a) Show that free-fall in the Earth's gravitational field from infinity results in the same velocity at the surface as if the object had dropped a distance R_{E} under a constant acceleration of g.

   (b) Show that for heights h small compared to R_{E} the first correction to the velocity at the surface for an object in free-fall is: $v = \sqrt{2gh}\left(1 - \frac{h}{2R_{\mathrm{E}}}\right)$.

(c) For a height h small compared to R_{E} show that the fractional change in period of a pendulum clock is given by: $\Delta T/T = h/R_{\mathrm{E}}$. Does the clock gain or lose and evaluate the change for a height of 100 m in seconds per week.

12. A canon is placed on an inclined plane of angle α with the horizontal. The gun is aligned with the steepest line of the plane and can fire shells in both the upward and downward directions, at an initial speed V and an angle θ with respect to the plane (upwards).

    (a) Find where the shell will land up the plane for $\theta = \alpha = 30°$.

    (b) Show that the ratio of the maximum ranges up and down the plane is given by:

$$\frac{R_{\text{up}}}{R_{\text{down}}} = \frac{1 - \sin\alpha}{1 + \sin\alpha}$$

13. A body of mass M, travelling in a straight horizontal line, is supplied with constant power P and is subjected to a resistance Mkv^2, where v is its speed and k is a constant.

    (a) Show that the speed of the body cannot exceed a certain value v_{m} and find an expression for v_{m}.

    (b) Show that, starting from rest, the body acquires half the maximum speed after travelling a distance $\ln(8/7)/(3k)$.

    (c) If the power is then cut off and an additional retarding force of constant value F imposed, find the subsequent time which elapses before the body comes to rest.

14. This question follows on from Question 10:

    (a) Calculate by direct integration the energy dissipation during the upward journey and show that your result is consistent with the work-energy theorem.

    (b) Using the result of Q10 (b) for the final velocity, show that for $v_0 = v_l$ half of the initial energy is lost during the round trip.