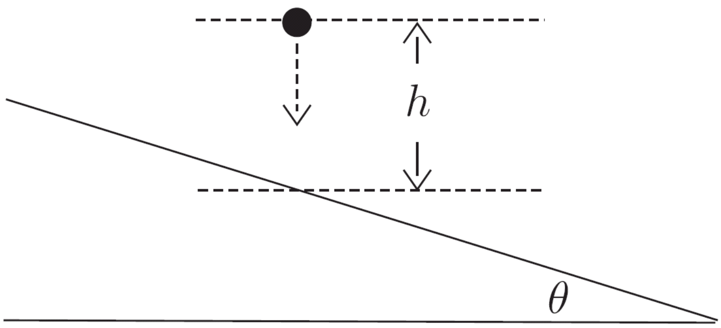
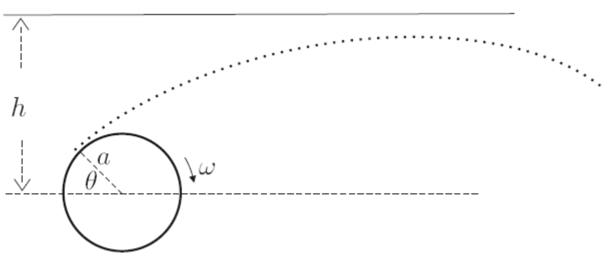
# Prelims: Mechanics III




1. A small sphere is released from rest and, after falling a vertical distance $h$ freely under gravity, bounces on a smooth inclined plane (at an angle $\theta < 90$ to the horizontal). Given that the sphere loses no energy on impact in what direction (relative to the upward normal to the plane) will it move immediately after impact? [Hint: choose axes parallel to and perpendicular to the plane.] Show that the distance down the plane between this impact and the next is $8h\sin\theta$.

2. Drops of water are thrown tangentially off the rim of a wheel of radius $a$, rotating in a vertical plane about its axis, which is fixed and horizontal, with angular velocity $\omega > \sqrt{g/a}$. Neglecting air resistance, determine the point on the wheel rim from which drops will rise to the maximum height above the wheel axis (consider the vertical contributions to PE and KE as a function of angle $\theta$). Hence show that if a horizontal ceiling, at height $h > a$ above the axis of the wheel, is not to be spattered with water, $\omega$ must satisfy

$$(a\omega)^2 \quad < \quad gh + g\sqrt{h^2 - a^2}.$$




3. Two masses $m$ and $\delta m$ ($\delta m \ll m$) are joined together, and moving with velocity $v$ in some inertial frame. At some instant the mass $\delta m$ flies apart from $m$ (e.g. because of the release of a spring) along the direction of motion with a relative velocity $w$ with respect to the mass $m$, such that the velocity of $m$ becomes $v + \delta v$. Show that, if the small mass is projected in the backwards direction,

$$m\delta v = w\delta m.$$

Explain whether the same expression applies for the case in which the small mass is projected in the forwards direction.

We now turn to the case of a rocket which burns fuel, so that the mass of the rocket remaining at time $t$ is $m(t)$. Show that the acceleration of the rocket is given by the equation

$$\frac{dv}{dt} = -\frac{V}{m}\frac{dm}{dt}$$

paying careful attention to signs. Finally, modify the rocket equation to take account of motion against a constant gravitational field (acceleration due to gravity = $-g$).

4. A rocket of initial mass $M$, of which half is fuel, is launched vertically upwards at $t = 0$. It burns the fuel at a mass rate of $\alpha$ (a positive quantity) and ejects it backwards from the rocket at a velocity $V$ with respect to the rocket.

   (a) Show that the equation governing the (vertical) speed v of the rocket is

   $$\frac{dv}{dt} = -g + \frac{\alpha V}{M - \alpha t}$$

   where $g$ is the acceleration due to gravity, assumed constant.

   (b) Show that the rocket cannot leave the launch-pad on ignition unless $\alpha V > Mg$.

   (c) Given $\alpha V > Mg$, show that fuel burn-out occurs at $t = M/2\alpha$.

   (d) Show that the upward velocity at burn-out is $v = V \ln 2 - gM/(2\alpha)$

   (e) Find the height of the rocket at burn-out and the maximum height reached.
   [You need $\int \ln x \, dx = x \ln x - x$.]

5. A rocket of total initial mass (casing + fuel + payload) $Nm$, where $m$ is the payload mass, ejects mass at a constant velocity $u$ relative to the rocket. The ratio of (casing mass) to (casing + fuel mass) is $r$. Ignore gravity and air-resistance and assume that the rocket starts from rest.

   (a) One stage rocket. Show that the payload achieves a final velocity of:

   $$v_1 = u \ln \left[ \frac{N}{rN + (1-r)} \right]$$

   (b) Two stage rocket. The total mass of both stages (fuel+casing+payload) is now $Nm$ and the total mass of the second stage (fuel+casing+payload) is $nm$. The other conditions for both stages are the same as above. Assuming that the first stage is dropped when its fuel is exhausted show that the final velocity of the payload is:

   $$v_2 = u \ln \left[ \frac{N}{rN + n(1-r)} \right] + u \ln \left[ \frac{n}{rn + (1-r)} \right]$$

   (c) If $N$ and $r$ are constant, show that $v_2$ is a maximum for $n^2 = N$ and then:

   $$v_2^{\max} = 2u \ln \left[ \frac{n}{rn + (1-r)} \right]$$

(d) Realistic maximum values of $u$ from chemical combustion are of the order of 2.9 km s$^{-1}$. If $r = 0.1$ could Earth escape speed be reached with either a one or two stage rocket?

6. A particle moves subject to a force field $\mathbf{F} = \alpha(2xy, x^2, 0)$, where $\alpha$ is a constant.

   The particle moves from the origin O to the point P with coordinates (1, 1, 0). What is the work done by **F** on the particle for each of the following paths from O to P:

   (a) a straight line from O to (1, 0, 0) followed by a straight line to P;

   (b) a straight line from O to (0, 1, 0) followed by a straight line to P;

   (c) a straight line direct from O to P.

   Repeat for the force field $\mathbf{G} = \alpha(x^2, 2xy, 0)$, and comment on the differences.

7. Consider the period of a simple pendulum with length $l$ and bob of mass $m$ moving under gravity, but not confined to small–angle oscillations. It was shown in Problem Set I that, on dimensional grounds, the period of oscillations is given by

$$T = kf(\theta_0)\sqrt{l/g},$$

   where $\theta$ is the angular amplitude, $k$ a dimensionless constant and $f$ an arbitrary function.

   (a) Re-work this problem by using energy conservation or otherwise to show that for small angle oscillations $kf(\theta_0) = 2\pi$. Show that for large oscillations $k = 2\sqrt{2}$ and

$$f(\theta_0) = \int_0^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\, d\theta.$$

   (b) Use the substitution: $\sin\frac{\theta}{2} = \sin\frac{\theta_0}{2}\sin\phi$ and $\alpha = \sin\frac{\theta_0}{2}$ to show that the period of oscillation is given by:

$$T = 4\sqrt{l/g}\int_0^{\pi/2} \frac{1}{\sqrt{1 - \alpha^2\sin^2\phi}}\, d\phi$$

   (c) Expand the integrand in the above expression to estimate the period of oscillation to second order in $\theta_0$. Estimate the percentage error if using the small angle approximation for $\theta_0 = \pi/3$.

8. The motion of a particle of mass $m$ and charge $q$ moving in a magnetic field $\mathbf{B} = (0, 0, B)$ is governed by Newton's Second Law in the form

$$m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}, \qquad \dagger$$

   where $\mathbf{r}(t) = (x(t), y(t), z(t))$ is the position of the particle at time $t$. At $t = 0$ the particle is at the origin and has velocity $(0, v_0, w_0)$.

Write down the three components of equation (†), and integrate them with respect to time, paying attention to the initial conditions. Show that

$$\ddot{y} + \omega^2 y \;=\; 0 \text{ where } \omega = qB/m,$$

and hence that $y = (v_0/\omega)\sin\omega t$. Find also $x$ and $z$, and describe and sketch the path of the particle.

Give a simple physical description of the balance of forces acting on the particle.

9. A passenger holding a parcel of mass $m$ is standing in a lift which is being accelerated upward by a constant force $F$. The total mass of lift plus passenger is $M$. (Assume $M \gg m$.)

   (a) What is the acceleration of the lift?

   (b) What weight does the parcel appear to have for the passenger?

   (c) If the passenger drops the parcel from height h, how long does it take to reach the floor? Try this both in the lift frame (non-inertial) and an inertial frame (eg w.r.t. lift shaft).

   (d) If the lift changes to uniform velocity, what must the value of $F$ become? Under these circumstances what is the answer to (c)? Comment on your result.

10. A particle P is moving along an arbitrary path C with velocity $\mathbf{v}(t)$ and position vector $\mathbf{r}(t)$ with respect to an origin O.

    (a) Show that if the particle velocity satisfies the relation $\dot{\mathbf{r}} = \mathbf{c} \times \mathbf{r}$ where $\mathbf{c}$ is a constant vector then P moves in uniform circular motion.

    (b) Let $\hat{\mathbf{u}}$ and $\hat{\mathbf{n}}$ be respectively the instantaneous tangential and normal unit vectors at point $x$, and $s$ is the (scalar) length along the path. Show that the derivative of one unit vector with respect to time or space ($s$) is proportional the other. Hence show that $\frac{d\hat{\mathbf{u}}}{ds} = \frac{1}{\rho}\hat{\mathbf{n}}$ where $\rho$ is the radius of curvature at $x$.

    (c) Use the relation $\mathbf{v} = v\hat{\mathbf{u}}$ to show that the acceleration of the particle is given by $\mathbf{a} = \dot{v}\hat{\mathbf{u}} + \frac{v^2}{\rho}\hat{\mathbf{n}}$

11. For a projectile with a linear air resistance force $F_s = -mkv$, show that the maximum horizontal range is given by the equation

$$\left(v_0 + \frac{g}{k}\right)\frac{x}{u_0} + \frac{g}{k^2}\ln\left(1 - \frac{kx}{u_0}\right) \;=\; 0$$

where $u_0 = V cos\theta$, $v_0 = V sin\theta$ are the horizontal and vertical components of the initial velocity.

The above equation cannot be solved in closed form, either it is solved numerically or it may be approximated, assuming that the correction with $k \neq 0$ is small. For the latter method show that

$$x_{\max} \;\approx\; \frac{2v_0u_0}{g} - \frac{8v_0^2u_0}{3g^2}k.$$