

Prelims: Mechanics IV

1. A particle of mass m is constrained to slide on the inside of a vertical smooth semi-circular ring of radius r . The position of the particle is described by a polar coordinate system whose origin is at the centre of the circle with axes along the orthogonal unit vectors \hat{r} and $\hat{\theta}$ where θ is the angle between the radius vector r and the vertical line that passes through the origin.

(a) Derive the general relationships for velocity and acceleration in polar coordinates. Hence show that the acceleration of the particle may be written as

$$\mathbf{a} = r\ddot{\theta}\hat{\theta} - \frac{v^2}{r}\hat{r}.$$

where $v = r\dot{\theta}$ is the magnitude of the particle velocity.

(b) Use the equation of motion in the $\hat{\theta}$ direction to calculate the period of small oscillation about the bottom point.

(c) Write down the resultant force acting on the particle as a function of θ . Assume the particle is released from rest at the top of the semi-circle, use conservation of energy and Newton's Second Law in the \hat{r} direction to calculate the reaction force exerted by the surface at $\theta = 60^\circ$.

2. A particle is moving under an attractive force $f(r)$ per unit mass directed along the line joining the particle to a fixed point O. The vector displacement of the particle from O at time t is \mathbf{r} . Show that $\mathbf{r} \times \dot{\mathbf{r}}$ is a constant vector and hence that the particle moves in a plane. Using plane polar coordinates (r, θ) for motion within the plane: define the angular momentum J ; using the transverse equation of motion show that $|J|$ is a constant; show that the radius vector r sweeps out area at a constant rate.

3. Two particles of mass m are connected by a light inextensible string of length l . One of the particles moves on a smooth horizontal table in which there is a small hole. The string passes through the hole so that the second particle hangs vertically below the hole.

(a) Use the conservation of energy and angular momentum to show that:

$$\dot{r}^2 = A - B/r^2 - gr$$

where $r(t)$ is the distance of the first particle from the hole, A and B are constants and g is the acceleration due to gravity. Given that the particle on the table at a distance $l/2$ from the hole moves with a speed v_0 directed perpendicular to the string, find the values of A and B .

(b) Show that the condition for the particle on the table to move in circular motion is given by $v_0^2/gl = 1/2$.

[Hint: use polar coordinates in the plane of the table, with the origin at the hole centre.]

4. (a) If a communication satellite is to remain in orbit constantly above a particular city on the equator, what distance above this city will the orbit be? Why must the city be on the equator?
- (b) A binary star consists of two stars bound together by gravity. From spectroscopic studies of Plaskett's binary it is known that the period of revolution of the stars about their centre of mass is 14.4 days, the speed of each component is 220 km s^{-1} and they are moving roughly in opposite directions along a nearly circular orbit. Show that the masses of the two component stars are roughly equal. Assuming a circular orbit, find the distance between the two stars and their masses.

5. A galaxy may be modelled as a dilute spherical distribution of stars with constant density within a radius R and total mass M . Find expressions for the velocity v of an individual star of mass m in a circular orbit about the centre of the galaxy at distances r both greater than and less than R .

[For the latter calculation you will need to use the fact that the gravitational force at an interior point r is given by the mass contained within a sphere of radius r .]

Sketch the dependence of v on r . Observations of $v(r)$ show that for many galaxies the measured curve requires a larger mass than that determined from the luminous matter. This is the so-called dark matter problem and it is a hot topic in current astrophysical research.

6. It is required to put a satellite into an orbit with apogee (furthest distance from the centre of the planet) of $5R/2$, where R is the radius of the planet. The satellite is to be launched from the surface with a speed v_0 at 30° to the local vertical. If M is the mass of the planet show that $v_0^2 = 5GM/4R$ (use conservation of energy and angular momentum). Assume that the planet is not rotating and that effects due to the planetary atmosphere can be ignored. (Adapted from French & Ebison)
7. An incoming comet of mass m is detected at the edge of the solar system. It has a velocity v and an impact parameter b (the minimum distance between the centre of the Sun and the extrapolation of the comet's initial path) with respect to the Sun (mass M).

- (a) Sketch the expected trajectory of the comet. Indicate the impact parameter b and the angle of scattering ϕ on your plot.
- (b) Show that the total energy of the comet as a function of position vector \mathbf{r} from the centre of the Sun may be written as:

$$E = \frac{1}{2}mv^2 + \frac{J^2}{2mr^2} - \frac{\alpha}{r}$$

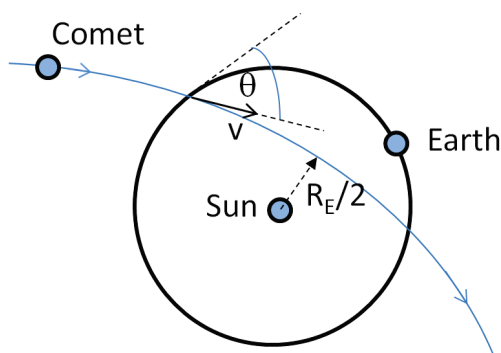
Find expressions for J and α .

- (c) Using the result in (b) or otherwise, show that the distance of closest approach to the centre of the Sun is

$$r_{\min} = \frac{GM}{v^2} \left[\sqrt{1 + \left(\frac{bv^2}{GM} \right)^2} - 1 \right]$$

where G is the gravitational constant.

8. The comet of Question 7 is deflected by the Sun and on its trajectory crosses the Earth's orbit. The distance of closest approach is measured to be half the distance between the Earth and the Sun, R_E , assuming the Earth's orbit to be circular. The velocity at that point is 60 km s^{-1} , which happens to be twice the orbital speed of the Earth. What is the comet's speed v when it crosses the Earth's orbit? What angle θ does the comet's trajectory make with the Earth's orbit at that point?



9. A satellite of mass m is put into a circular orbit of radius r_0 around the Earth (of mass M).
- (a) Given that the speed of the satellite is v_0 , show that

$$r_0 = \frac{GM}{v_0^2}$$

where G is the gravitational constant.

- (b) Given that J_0 is the satellite's angular momentum and E_0 its total energy (kinetic plus potential), show that

$$J_0 = \frac{GMm}{v_0}$$

and

$$E_0 = -\frac{1}{2}mv_0^2$$

- (c) Whilst in its circular orbit, the satellite is given an instantaneous impulse which changes its angular momentum from J_0 to αJ_0 , where $\alpha < 1$. The satellite's energy remains unchanged. Describe the shape of the new orbit and, using the results of Question 7(b), calculate the minimum and maximum values for the distance between the satellite and the centre of the Earth if $\alpha = 12/13$.
- (d) Assuming that whilst the satellite in its circular orbit the impulse leaves the angular momentum unchanged but increases its kinetic energy by 1%. Find the difference between the maximum and minimum distances from the centre of the earth. Sketch the orbits of (c) and (d) comparing with the original circular orbit.

10. A particle of electric charge q_1 and mass m is projected, with velocity v_0 and impact parameter d ($d > 0$), towards a fixed nucleus of opposite charge q_2 (attractive force). Show that the angle of deflection α is given by

$$\cot\left(\frac{\alpha}{2}\right) = \frac{4\pi\epsilon_0}{q_1q_2}dv_0^2m.$$

Sketch the trajectory.

By making the replacement $\frac{q_1q_2}{4\pi\epsilon_0} \rightarrow GMm$ show that the above result may be adapted to find the angle through which the path of a comet of mass m is ultimately deflected as it passes the Sun.

11. Consider an isolated two-body system of masses m_1 and m_2 moving under their internal gravitational interaction. The particles can be treated as point masses and their position in an inertial frame is described by the vectors \mathbf{r}_1 and \mathbf{r}_2 .

(a) Show that the EOM for either particle may be written as

$$-\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} = \mu\ddot{\mathbf{r}}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ is the relative position vector and $\mu = \frac{m_1m_2}{m_1+m_2}$ is the reduced mass.

- (b) Show that if the motion is viewed in the ZMF, the kinetic energy of the two particles is given by $T = (1/2)\mu\dot{\mathbf{r}}^2$
- (c) Conclude therefore that the motion of the two particles may be analysed using an equivalent system comprising a *single particle* of mass μ moving under the influence of an attractive central force $f(r)\hat{\mathbf{r}}$ where \mathbf{r} is the position vector of the particle with respect to the force centre. Write down the total energy and the effective potential of the equivalent system.
12. A particle of mass m moves in the $(x - y)$ plane under the influence of a spring-like force of the form $F = -kr$, where $k > 0$ is a positive constant. Assuming that at $t = 0$ the position of the particle is given by $r_0 = (x_0, 0)$ and the velocity by $v_0 = (0, u_0)$,
- (a) write the EOM in Cartesian coordinates and show that the path of motion is an ellipse centred at the origin of coordinates. Find the major and the minor axes of the ellipse.
- (b) Derive an expression for the effective potential and use it to analyse the motion and to show that it is always bound.
13. The two individual stars in a binary system ($m_1 = m$, $m_2 = 2m$) are in circular orbit about their common centre of mass and are separated by a distance a . At some stage, the more massive star explodes and as a result suffers a spherically symmetric loss of mass. After the explosion the masses of the stars become equal:
- (a) using circular motion dynamics, calculate the total energy, angular momentum and period of the binary star as it is viewed in the ZMF (before the explosion).

- (b) Use the method described in problem (9) below to calculate the total energy and period of the equivalent single particle system orbit, before the explosion.
 - (c) Using the equivalent single particle system or otherwise, show that the binary star will remain intact (bound) after the explosion.
 - (d) Sketch the equivalent single particle orbit and the actual binary orbits as viewed in the new ZMF (after the explosion).
14. An efficient way to reach the Moon is to first put the spacecraft in a low circular Earth orbit (radius r_0 , speed v_0). The speed is then boosted to v_p giving an elliptical orbit with apogee at the Moon's orbit, r_a , and perigee at r_0 . Show that

$$\left(\frac{v_p}{v_0}\right)^2 = \frac{2r_a}{r_0 + r_a}.$$

Taking $r_0 \sim R_e$, $r_a \sim 60R_e$ calculate v_0 and v_p . Estimate by how much the Moon's orbit will be missed if v_p/v_0 is incorrect by 0.1 %. What other considerations are necessary for a successful rendezvous between spacecraft and Moon? [$R_e = 6.38 \times 10^6$ m] (Adapted from Fowles & Cassidy.)