Prelims: Mechanics VI

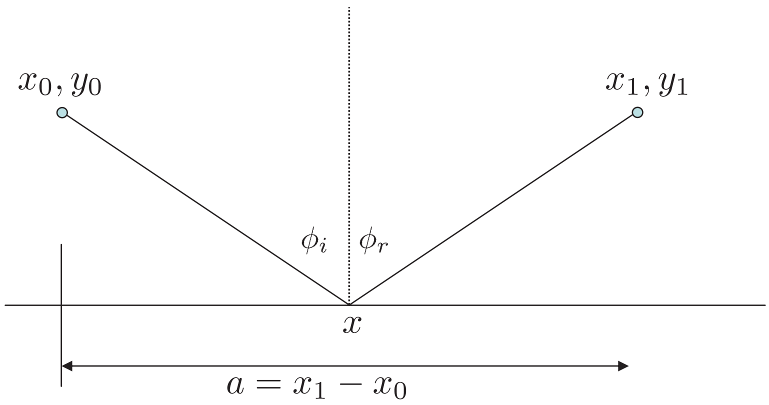
1. A light beam is propagating in the $x - y$ plane in a media whose refraction index n depends only on y.

   (a) Use Fermat's principle to show that the trajectory of the beam from (x_0, y_0) to (x_1, y_1) may be obtained by minimizing the functional

   $$S(y) \;=\; c^{-1}\int_{x_0}^{x_1} n(y)\sqrt{1+y'^2}\,dx$$

   where $y' = dy/dx$ and c is the speed of light in vacuum.

   (b) Now let n be independent of x and y.

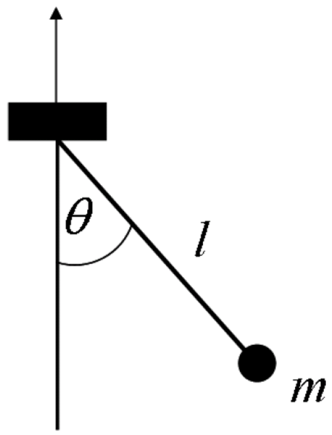
   

   A light ray propagates from (x_0, y_0) to (x_1, y_1) by reflection from the surface of a flat mirror located at in the plane $y = 0$ as shown in Fig 1. Show that the angle of reflection ϕ_r is equal to the angle of incidence ϕ_i

2. Consider a particle of mass m moving in the x, y plane under the influence of the potential $V(r)$ where **r** is the position vector of the particle in an inertial reference frame.

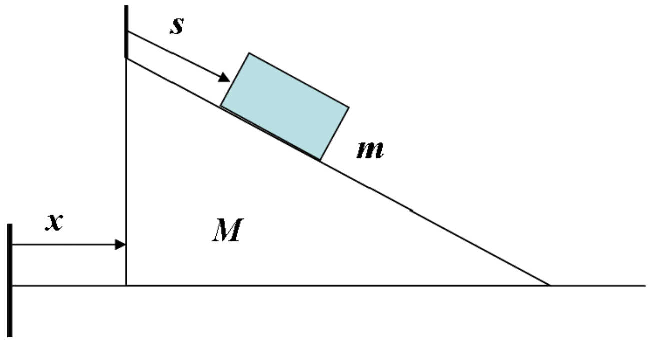
   Construct the Lagrangian and the Hamiltonian of the particle in polar polar coordinates r, θ, hence find which quantities are constants of motion. Is this consistent with what you expected from Newtonian mechanics?

3.



Use E-L equation to calculate the period of oscillation of a simple pendulum of length l and bob mass m in the small angle approximation. Assume now that the pendulum support is accelerated in the vertical direction at a rate a, find the period of oscillation. For what value of a does the pendulum not oscillate? Comment on this result.

4.

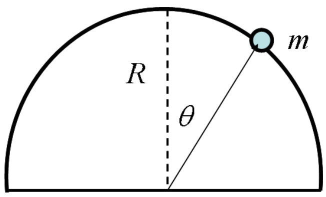


A block of mass m slides on a frictionless inclined plane of mass M, which itself rests on a horizontal frictionless surface.

(a) Choose the displacement of the inclined plane x and the displacement of the block m relative to the inclined plane s as generalized coordinates and find the Lagrangian of the system.

(b) Write down the E-L equation for each coordinate and find the acceleration of the inclined plane.

5.



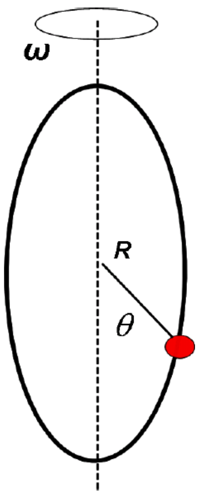
A particle of mass m slides without friction down the surface of a hemisphere of radius R.

(a) Construct the Lagrangian of the problem in terms of the polar coordinates (r, θ), in the range when the constraint $r = R$ is valid. Find the equation of motion.

(b) Allow the radius of the sphere to vary by an infinitesimal amount and write the equation of motion with r as a free variable. Include the potential $V(r)$ of the reaction force applied by the hemisphere on the object. Write the new Lagrangian and find the reaction force for $r = R$. Compare with the result derived from NII.

(c) Assuming that the particle is released from the top of the sphere from rest, show that the particle leaves the surface at an angle $\cos\theta_{\max} = 2/3$.

6.



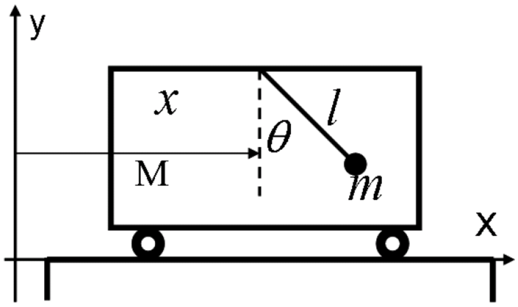
A vertical circular hoop of radius R rotates about a vertical axis at an angular velocity ω. A bead of mass m can slide on the hoop without friction and is constrained to stay on the hoop. By taking the angle θ between the radius line and the vertical, as a generalized coordinate,

(a) Find the Lagrangian and the equation of motion. Using the concept of effective potential or otherwise, find the three equilibrium positions of the bead.

(b) Discuss the stability of each equilibrium point and find the frequency of small oscillations about the stable ones.

(c) Find the Hamiltonian and the total energy $T + V$. Is either of them a constant of motion?

7.



A box of mass M can slide horizontally on a frictionless surface. A simple pendulum of string length l and mass m, is suspended inside the block. Denote the coordinate of the centre of mass of the box by x and the angle that the pendulum makes with the vertical by θ. At $t = 0$ the pendulum displacement is $\theta = \theta_0 \neq 0$

(a) Find the Lagrangian and the equation of motion for the generalized coordinates x and θ. Which conservation law is obtained as a result of the cyclic coordinate?

(b) Find the solutions for x and θ in the small angle approximation, hence show that the pendulum and the box execute SHO about their centre of mass at a frequency

$$\omega = \left[\frac{m+M}{M}\right]^{\frac{1}{2}} \left(\frac{g}{l}\right)^{\frac{1}{2}}$$

8. A small uniform cylinder of radius a rolls without slipping on the inside of a large, fixed cylinder of radius b ($b \geq a$). Show that the period of small oscillations of the rolling cylinder is that of a simple pendulum of length $3(b-a)/2$.

9. A ladder of length l stands on a frictionless floor and leans on a frictionless wall. The ladder is allowed to slide while remaining supported by the floor and wall, and its inclination relative to the vertical wall is given by the angle θ.

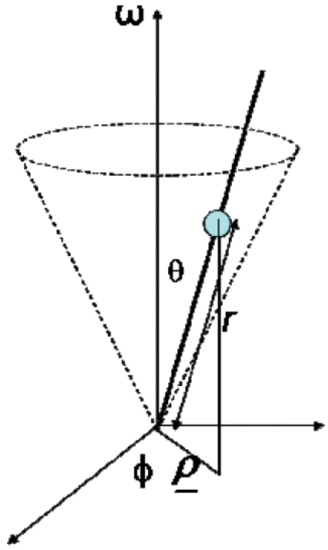
(a) Write down the kinetic and potential energy of the ladder.

(b) Show that the constraints allow you to write the Lagrangian in a single coordinate θ.

(c) Use the Lagrangian to discuss the conservation of the total energy E, the Hamiltonian H and angular momentum J. Write down the EOM using E-L equation.

[The moment of inertia if the ladder about its centre of mass is $I = \frac{1}{12}ml^2$]

10.



A bead of mass m is constrained to slide on a frictionless wire which is made to rotate about a vertical axis at an angular velocity ω. The wire is tilted away from the vertical by a fixed angle θ and the location of the bead is measured by the coordinate r.

(a) Write down the equation of motion of the bead using the E-L equation. Test the integrity of your equation by taking extreme values of θ.

(b) Find the general solution assuming that at $t = 0$, $r = r_0$, $\dot{r} = 0$ and using the definition $r_1 = g \cos \alpha / \omega^2 \sin^2 \alpha$. Based on this solution, show that for $r_0 = r_1$, the bead moves in circular motion (as expected!). Describe the motion for $r_0 < r_1$ and $r_0 > r_1$.

(c) Which of the following quantities is a constant of the bead motion: angular momentum with respect to the origin, the Hamiltonian, total energy?